In 1831 Michael Faraday discovered that when a wire passes by a magnet, a small electric current is produced in the wire. Now we generate massive amounts of electricity by simultaneously rotating thousands of wires near large electromagnets. Because electric current alternates its direction on electrical wires, it is modeled accurately by either the sine or the cosine function.

We give many examples of applications of the trigonometric functions to electricity and other phenomena in the examples and exercises in this chapter, including a model of the wattage consumption of a toaster in Section 7.4, Example 5.
7.1 Fundamental Identities

**Negative-Angle Identities**  As suggested by the circle shown in Figure 1, an angle \( \theta \) having the point \((x, y)\) on its terminal side has a corresponding angle \(-\theta\) with the point \((x, -y)\) on its terminal side. From the definition of sine,

\[
\sin(-\theta) = -\frac{y}{r} \quad \text{and} \quad \sin \theta = \frac{y}{r}, \quad (\text{Section 5.2})
\]

so \( \sin(-\theta) \) and \( \sin \theta \) are negatives of each other, or

\[
\sin(-\theta) = -\sin \theta.
\]

Figure 1 shows an angle \( \theta \) in quadrant II, but the same result holds for \( \theta \) in any quadrant. Also, by definition,

\[
\cos(-\theta) = \frac{x}{r} \quad \text{and} \quad \cos \theta = \frac{x}{r}, \quad (\text{Section 5.2})
\]

so

\[
\cos(-\theta) = \cos \theta.
\]

We can use these identities for \( \sin(-\theta) \) and \( \cos(-\theta) \) to find \( \tan(-\theta) \) in terms of \( \tan \theta \):

\[
\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin \theta}{\cos \theta} = -\tan \theta.
\]

Similar reasoning gives the following identities.

\[
csc(-\theta) = -\csc \theta, \quad \sec(-\theta) = \sec \theta, \quad \cot(-\theta) = -\cot \theta
\]

This group of identities is known as the **negative-angle** or **negative-number identities**.

**Fundamental Identities**  In Chapter 5 we used the definitions of the trigonometric functions to derive the reciprocal, quotient, and Pythagorean identities. Together with the negative-angle identities, these are called the **fundamental identities**.

### Fundamental Identities

#### Reciprocal Identities

\[
\cot \theta = \frac{1}{\tan \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}
\]

#### Quotient Identities

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}
\]

(continued)
7.1 Fundamental Identities

Teaching Tip. Encourage students to memorize the identities presented in this section as well as subsequent sections. Point out that numerical values can be used to help check whether or not an identity was recalled correctly.

Pythagorean Identities
\[
\sin^2 \theta + \cos^2 \theta = 1 \quad \text{tan}^2 \theta + 1 = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta
\]

Negative-Angle Identities
\[
\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta \quad \tan(-\theta) = -\tan \theta \\
\csc(-\theta) = -\csc \theta \quad \sec(-\theta) = \sec \theta \quad \cot(-\theta) = -\cot \theta
\]

NOTE The most commonly recognized forms of the fundamental identities are given above. Throughout this chapter you must also recognize alternative forms of these identities. For example, two other forms of \( \sin^2 \theta + \cos^2 \theta = 1 \) are
\[
\sin^2 \theta = 1 - \cos^2 \theta \quad \text{and} \quad \cos^2 \theta = 1 - \sin^2 \theta.
\]

Using the Fundamental Identities One way we use these identities is to find the values of other trigonometric functions from the value of a given trigonometric function. Although we could find such values using a right triangle, this is a good way to practice using the fundamental identities.

EXAMPLE 1 Finding Trigonometric Function Values Given One Value and the Quadrant

If \( \tan \theta = -\frac{5}{3} \) and \( \theta \) is in quadrant II, find each function value.
(a) \( \sec \theta \) \quad (b) \( \sin \theta \) \quad (c) \( \cot(-\theta) \)

Solution
(a) Look for an identity that relates tangent and secant.
\[
\tan^2 \theta + 1 = \sec^2 \theta \quad \text{Pythagorean identity}
\]
\[
\left(-\frac{5}{3}\right)^2 + 1 = \sec^2 \theta \\
\frac{25}{9} + 1 = \sec^2 \theta \\
\frac{34}{9} = \sec^2 \theta \quad \text{Combine terms.}
\]
\[
-\sqrt{\frac{34}{9}} = \sec \theta \quad \text{Take the negative square root. (Section 1.4)}
\]
\[
-\frac{\sqrt{34}}{3} = \sec \theta \quad \text{Simplify the radical. (Section R.7)}
\]

We chose the negative square root since \( \sec \theta \) is negative in quadrant II.

Teaching Tip. Warn students that the given information in Example 1, \( \tan \theta = -\frac{5}{3} = \frac{-5}{3} \), does not mean that \( \sin \theta = -5 \) and \( \cos \theta = 3 \). Ask them why these values cannot be correct.
(b) \[ \tan \theta = \frac{\sin \theta}{\cos \theta} \] Quotient identity

\[ \cos \theta \tan \theta = \sin \theta \] Multiply by \( \cos \theta \).

\[ \left( \frac{1}{\sec \theta} \right) \tan \theta = \sin \theta \] Reciprocal identity

\[ \left( -\frac{3\sqrt{34}}{34} \right) \left( -\frac{5}{3} \right) = \sin \theta \] From part (a), \( \frac{1}{\sec \theta} = -\frac{3\sqrt{34}}{34} = -\frac{3\sqrt{34}}{34} \),

\[ \tan \theta = -\frac{5}{3} \]

\[ \sin \theta = \frac{5\sqrt{34}}{34} \]

(c) \[ \cot(-\theta) = \frac{1}{\tan(-\theta)} \] Reciprocal identity

\[ \cot(-\theta) = \frac{1}{-\tan \theta} \] Negative-angle identity

\[ \cot(-\theta) = \frac{1}{-(-\frac{5}{3})} = \frac{3}{5} \] \( \tan \theta = -\frac{5}{3} \); Simplify. (Section R.5)

Now try Exercises 5, 7, and 9.

**CAUTION** To avoid a common error, when taking the square root, be sure to choose the sign based on the quadrant of \( \theta \) and the function being evaluated.

Any trigonometric function of a number or angle can be expressed in terms of any other function.

**EXAMPLE 2** Expressing One Function in Terms of Another

Express \( \cos x \) in terms of \( \tan x \).

**Solution** Since \( \sec x \) is related to both \( \cos x \) and \( \tan x \) by identities, start with \( 1 + \tan^2 x = \sec^2 x \).

\[ \frac{1}{1 + \tan^2 x} = \frac{1}{\sec^2 x} \] Take reciprocals.

\[ \frac{1}{1 + \tan^2 x} = \cos^2 x \] Reciprocal identity

\[ \pm \sqrt{\frac{1}{1 + \tan^2 x}} = \cos x \] Take square roots.

\[ \cos x = \frac{\pm 1}{\sqrt{1 + \tan^2 x}} \] Quotient rule (Section R.7); rewrite.

\[ \cos x = \frac{\pm \sqrt{1 + \tan^2 x}}{1 + \tan^2 x} \] Rationalize the denominator. (Section R.7)

Choose the + sign or the − sign, depending on the quadrant of \( x \).

Now try Exercise 43.
7.1 Fundamental Identities

We can use a graphing calculator to decide whether two functions are identical. See Figure 2, which supports the identity \( \sin^2 x + \cos^2 x = 1 \). With an identity, you should see no difference in the two graphs. 

All other trigonometric functions can easily be expressed in terms of \( \sin \theta \) and/or \( \cos \theta \). We often make such substitutions in an expression to simplify it.

**EXAMPLE 3  Rewriting an Expression in Terms of Sine and Cosine**

Write \( \tan \theta + \cot \theta \) in terms of \( \sin \theta \) and \( \cos \theta \), and then simplify the expression.

**Solution**

\[
\tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta}
\]

Pythagorean identity

**CAUTION** When working with trigonometric expressions and identities, be sure to write the argument of the function. For example, we would not write \( \sin^2 + \cos^2 = 1 \); an argument such as \( \theta \) is necessary in this identity.

---

### 7.1 Exercises

1. -2.6  2. -6.5  3. 6.25
4. -0.75  5. \( \sqrt{7} / 4 \)  6. -3\( \sqrt{10} / 10 \)
7. -2\( \sqrt{5} / 5 \)  8. -\( \sqrt{77} / 11 \)
9. -\( \sqrt{105} / 11 \)  10. -5/8

**Concept Check** Fill in the blanks.

1. If \( \tan x = 2.6 \), then \( \tan(-x) = \) ______.
2. If \( \cos x = -0.65 \), then \( \cos(-x) = \) ______.
3. If \( \tan x = 1.6 \), then \( \cot x = \) ______.
4. If \( \cos x = .8 \) and \( \sin x = .6 \), then \( \tan(-x) = \) ______.

Find \( \sin s \). See Example 1.

5. \( \cos s = \frac{3}{4} \), \( s \) in quadrant I  6. \( \cot s = -\frac{1}{3} \), \( s \) in quadrant IV
7. \( \cos(-s) = \frac{\sqrt{5}}{5} \), \( \tan s < 0 \)  8. \( \tan s = -\frac{\sqrt{7}}{2} \), \( \sec s > 0 \)
9. \( \sec s = \frac{11}{4} \), \( \tan s < 0 \)  10. \( \csc s = -\frac{8}{5} \)

11. Why is it unnecessary to give the quadrant of \( s \) in Exercise 10?
12. \(-\sin x\)  
13. odd  
14. \(\cos x\)  
15. even  
16. \(-\tan x\)  
17. odd  
18. \(f(-x) = f(x)\)  
19. \(f(-x) = -f(x)\)  
20. \(f(-x) = -f(x)\)  
21. \(\cos \theta = \frac{-\sqrt{5}}{3}\);  
\(\tan \theta = \frac{-2\sqrt{3}}{5}\); \(\cot \theta = \frac{-\sqrt{5}}{2}\);  
\(\sec \theta = \frac{-3\sqrt{3}}{5}\); \(\csc \theta = \frac{3}{2}\);  
22. \(\sin \theta = \frac{2\sqrt{6}}{5}\); \(\tan \theta = 2\sqrt{6}\);  
\(\cot \theta = \frac{\sqrt{6}}{12}\); \(\sec \theta = 5\);  
\(\csc \theta = \frac{5\sqrt{6}}{12}\);  
23. \(\sin \theta = -\frac{\sqrt{17}}{17}\);  
\(\cos \theta = \frac{4\sqrt{17}}{17}\); \(\cot \theta = -4\);  
\(\sec \theta = -\frac{\sqrt{17}}{4}\); \(\csc \theta = -\sqrt{17}\);  
24. \(\sin \theta = -\frac{2}{5}\);  
\(\cos \theta = -\frac{\sqrt{21}}{5}\); \(\tan \theta = \frac{2\sqrt{21}}{21}\);  
\(\cot \theta = \frac{\sqrt{21}}{2}\); \(\sec \theta = -\frac{5\sqrt{21}}{21}\);  
25. \(\sin \theta = \frac{3}{5}\); \(\cos \theta = \frac{4}{5}\);  
\(\tan \theta = \frac{3}{4}\); \(\sec \theta = \frac{5}{4}\); \(\csc \theta = \frac{5}{3}\);  
26. \(\cos \theta = -\frac{3}{5}\); \(\tan \theta = \frac{4}{3}\);  
\(\cot \theta = \frac{3}{4}\); \(\sec \theta = -\frac{5}{3}\);  
\(\csc \theta = -\frac{5}{4}\);  
27. \(\sin \theta = -\frac{\sqrt{7}}{4}\); \(\cos \theta = \frac{3}{4}\);  
\(\tan \theta = -\frac{\sqrt{7}}{3}\); \(\cot \theta = -\frac{3\sqrt{7}}{7}\);  
\(\csc \theta = -\frac{4\sqrt{7}}{7}\)  
28. \(\cos \theta = \frac{1}{5}\), \(\theta\) in quadrant I  
29. \(\sec \theta = -\frac{5}{2}\), \(\theta\) in quadrant III  
30. \(\cot \theta = \frac{4}{3}\), \(\sin \theta > 0\)  
31. \(\sin \theta = -\frac{4}{5}\), \(\cos \theta < 0\)  
32. \(\cos \theta = \frac{1}{4}\), \(\sin \theta > 0\)
28. \( \sin \theta = \frac{\sqrt{15}}{4} \); 
\( \tan \theta = -\sqrt{15} \); \( \cot \theta = -\frac{\sqrt{15}}{15} \);

\( \sec \theta = -4 \); \( \csc \theta = \frac{4\sqrt{15}}{15} \)

37. D 38. B

41. \( \sin \theta = \frac{\pm \sqrt{2x + 1}}{x + 1} \)

42. \( \tan \alpha = \frac{\pm 2\sqrt{2p + 4}}{p} \)

43. \( \sin x = \pm \sqrt{1 - \cos^2 x} \)

44. \( \cot x = \pm \sqrt{1 - \sin^2 x} \)

45. \( \tan x = \pm \sqrt{\sec^2 x - 1} \)

46. \( \cot x = \pm \sqrt{\csc^2 x - 1} \)

47. \( \csc x = \pm \sqrt{1 - \cos^2 x} \)

48. \( \sec x = \pm \sqrt{1 - \sin^2 x} \)

49. \( \cos \theta = 0.1 \)
50. \( \cot \theta = 0.51 \)
52. \( \csc^2 \theta = 0.53 \)
53. \( \cos^2 \theta = 0.54 \)
55. \( \sec \theta - \cos \theta = 0.56 \)
56. \( 1 + \cot \theta = 0.57 \)
58. \( \sin^2 \theta \cos^2 \theta = 0.59 \)
59. \( \tan \theta - \cot \theta = 0.60 \)
60. \( \cot^2 \theta = 0.61 \)

62. \( \tan^2 \theta = 0.63 \)

**Concept Check** For each expression in Column I, choose the expression from Column II that completes an identity.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\cos x}{\sin x} )</td>
<td>A. ( \sin^2 x + \cos^2 x )</td>
</tr>
<tr>
<td>( \tan x )</td>
<td>B. ( \cot x )</td>
</tr>
<tr>
<td>( \cos(-x) )</td>
<td>C. ( \sec^2 x )</td>
</tr>
<tr>
<td>( \tan^2 x + 1 )</td>
<td>D. ( \frac{\sin x}{\cos x} )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>E. ( \cos x )</td>
</tr>
</tbody>
</table>

**Concept Check** For each expression in Column I, choose the expression from Column II that completes an identity. You may have to rewrite one or both expressions.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\tan x \cos x )</td>
<td>A. ( \frac{\sin^2 x}{\cos^2 x} )</td>
</tr>
<tr>
<td>( \sec^2 x - 1 )</td>
<td>B. ( \frac{1}{\sec x} )</td>
</tr>
<tr>
<td>( \frac{\sec x}{\csc x} )</td>
<td>C. ( \sin(-x) )</td>
</tr>
<tr>
<td>( 1 + \sin^2 x )</td>
<td>D. ( \csc^2 x - \cot^2 x + \sin^2 x )</td>
</tr>
<tr>
<td>( \cos^2 x )</td>
<td>E. ( \tan x )</td>
</tr>
</tbody>
</table>

39. A student writes “1 + \cot^2 = \csc^2.” Comment on this student’s work.

40. Another student makes the following claim: “Since \( \sin^2 \theta + \cos^2 \theta = 1 \), I should be able to also say that \( \sin \theta + \cos \theta = 1 \) if I take the square root of both sides.” Comment on this student’s statement.

41. **Concept Check** Suppose that \( \cos \theta = \frac{1}{\sqrt{x + 1}} \). Find \( \sin \theta \).

42. **Concept Check** Find \( \tan \alpha \) if \( \sec \alpha = \frac{p + 4}{p} \).

**Write the first trigonometric function in terms of the second trigonometric function.** See Example 2.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin x )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>( \sec x )</td>
<td>( \tan x )</td>
</tr>
</tbody>
</table>

43. \( \sin x \); \( \cos x \)
44. \( \cot x \); \( \sin x \)
45. \( \tan x \); \( \sec x \)

46. **Write each expression in terms of sine and cosine, and simplify it.** See Example 3.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cot \theta \sin \theta )</td>
<td>( \sec \theta \cot \theta \sin \theta )</td>
</tr>
<tr>
<td>( \cos \theta \csc \theta )</td>
<td>( \sin \theta \cot \theta )</td>
</tr>
<tr>
<td>( \cos^2 \theta(1 + \tan^2 \theta) )</td>
<td>( \sin \theta \cot \theta )</td>
</tr>
<tr>
<td>( \cos \theta + \sin \theta )</td>
<td>( \sin \theta \cot \theta )</td>
</tr>
<tr>
<td>( 1 - \sin \theta )</td>
<td>( \sin \theta \cot \theta )</td>
</tr>
<tr>
<td>( 1 + \cot^2 \theta )</td>
<td>( \sin \theta \cot \theta )</td>
</tr>
<tr>
<td>( \sec \theta - \cos \theta )</td>
<td>( \sin \theta \cot \theta )</td>
</tr>
<tr>
<td>( \sin \theta \csc \theta - \sin \theta )</td>
<td>( \sin \theta \cot \theta )</td>
</tr>
<tr>
<td>( 1 + \tan^2 \theta )</td>
<td>( \sin \theta \cot \theta )</td>
</tr>
</tbody>
</table>
63. \( \sec^2 \theta \)
64. \(-\sin \theta \)

65. \( \frac{25\sqrt{6} - 60}{12} = -\frac{25\sqrt{6} - 60}{12} \)
66. \( \frac{2\sqrt{2} + 8}{9} = -\frac{2\sqrt{2} + 8}{9} \)

67. \(-\sin(2x)\)
68. It is the negative of \(\sin(2x)\).
69. \(\cos(4x)\)
70. It is the same function.
71. (a) \(-\sin(4x)\) (b) \(\cos(2x)\) (c) \(5 \sin(3x)\)
72. identity 73. not an identity 74. not an identity 75. identity 76. not an identity

63. \( \sin^2 \theta + \tan^2 \theta + \cos^2 \theta \)
64. \( \frac{\tan(-\theta)}{\sec \theta} \)

65. Concept Check Let \( \cos x = \frac{1}{2} \). Find all possible values for \( \frac{\sec x - \tan x}{\sin x} \).
66. Concept Check Let \( \csc x = -3 \). Find all possible values for \( \frac{\sin x + \cos x}{\sec x} \).

**Relating Concepts**

For individual or collaborative investigation
(Exercises 67–71)

In Chapter 6 we graphed functions defined by

\[ y = c + a \cdot f[b(x - d)] \]

with the assumption that \( b > 0 \). To see what happens when \( b < 0 \), work Exercises 67–71 in order.

67. Use a negative-angle identity to write \( y = \sin(-2x) \) as a function of \( 2x \).
68. How does your answer to Exercise 67 relate to \( y = \sin(2x) \)?
69. Use a negative-angle identity to write \( y = \cos(-4x) \) as a function of \( 4x \).
70. How does your answer to Exercise 69 relate to \( y = \cos(4x) \)?
71. Use your results from Exercises 67–70 to rewrite the following with a positive value of \( b \).
   (a) \( \sin(-4x) \) (b) \( \cos(-2x) \) (c) \( -5 \sin(-3x) \)

Use a graphing calculator to decide whether each equation is an identity. (Hint: In Exercise 76, graph the function of \( x \) for a few different values of \( y \) (in radians).)

72. \( \cos 2x = 1 - 2 \sin^2 x \)
73. \( 2 \sin s = \sin 2s \)
74. \( \sin x = \sqrt{1 - \cos^2 x} \)
75. \( \cos 2x = \cos^2 x - \sin^2 x \)
76. \( \cos(x - y) = \cos x - \cos y \)

**7.2 Verifying Trigonometric Identities**

Verifying Identities by Working with One Side  •  Verifying Identities by Working with Both Sides

Recall that an identity is an equation that is satisfied for all meaningful replacements of the variable. One of the skills required for more advanced work in mathematics, especially in calculus, is the ability to use identities to write expressions in alternative forms. We develop this skill by using the fundamental identities to verify that a trigonometric equation is an identity (for those values of the variable for which it is defined). Here are some hints to help you get started.
7.2 Verifying Trigonometric Identities

Hints for Verifying Identities

1. Learn the fundamental identities given in the last section. Whenever you see either side of a fundamental identity, the other side should come to mind. Also, be aware of equivalent forms of the fundamental identities. For example, \( \sin^2 \theta = 1 - \cos^2 \theta \) is an alternative form of the identity \( \sin^2 \theta + \cos^2 \theta = 1 \).

2. Try to rewrite the more complicated side of the equation so that it is identical to the simpler side.

3. It is sometimes helpful to express all trigonometric functions in the equation in terms of sine and cosine and then simplify the result.

4. Usually, any factoring or indicated algebraic operations should be performed. For example, the expression \( \sin^2 x + 2 \sin x + 1 \) can be factored as \( (\sin x + 1)^2 \). The sum or difference of two trigonometric expressions, such as \( \frac{1}{\sin \theta} + \frac{1}{\cos \theta} \), can be added or subtracted in the same way as any other rational expression.

5. As you select substitutions, keep in mind the side you are not changing, because it represents your goal. For example, to verify the identity

\[
\tan^2 x + 1 = \frac{1}{\cos^2 x},
\]

try to think of an identity that relates \( \tan x \) to \( \cos x \). In this case, since \( \sec x = \frac{1}{\cos x} \) and \( \sec^2 x = \tan^2 x + 1 \), the secant function is the best link between the two sides.

6. If an expression contains \( 1 + \sin x \), multiplying both numerator and denominator by \( 1 - \sin x \) would give \( 1 - \sin^2 x \), which could be replaced with \( \cos^2 x \). Similar results for \( 1 - \sin x \), \( 1 + \cos x \), and \( 1 - \cos x \) may be useful.

Caution: Verifying identities is not the same as solving equations. Techniques used in solving equations, such as adding the same terms to both sides, or multiplying both sides by the same term, should not be used when working with identities since you are starting with a statement (to be verified) that may not be true.

Verifying Identities by Working with One Side To avoid the temptation to use algebraic properties of equations to verify identities, work with only one side and rewrite it to match the other side, as shown in Examples 1–4.
EXAMPLE 1  Verifying an Identity (Working with One Side)

Verify that the following equation is an identity.

\[
\cot s + 1 = \csc s (\cos s + \sin s)
\]

**Solution**  We use the fundamental identities from Section 7.1 to rewrite one side of the equation so that it is identical to the other side. Since the right side is more complicated, we work with it, using the third hint to change all functions to sine or cosine.

**Steps**

<table>
<thead>
<tr>
<th>Right side of given equation</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\csc s (\cos s + \sin s))</td>
<td>(= \frac{1}{\sin s} (\cos s + \sin s))</td>
</tr>
<tr>
<td></td>
<td>(= \frac{\cos s}{\sin s} + \frac{\sin s}{\sin s})</td>
</tr>
<tr>
<td></td>
<td>(= \cot s + 1)</td>
</tr>
<tr>
<td>Left side of given equation</td>
<td>(\cot s + 1)</td>
</tr>
</tbody>
</table>

The given equation is an identity since the right side equals the left side.

Now try Exercise 33.

EXAMPLE 2  Verifying an Identity (Working with One Side)

Verify that the following equation is an identity.

\[
\tan^2 x(1 + \cot^2 x) = \frac{1}{1 - \sin^2 x}
\]

**Solution**  We work with the more complicated left side, as suggested in the second hint. Again, we use the fundamental identities from Section 7.1.

\[
\tan^2 x(1 + \cot^2 x) = \tan^2 x + \tan^2 x \cot^2 x
\]

\[
= \tan^2 x + \frac{1}{\tan^2 x} \cot^2 x
\]

\[
= \tan^2 x + \frac{1}{\tan^2 x}
\]

\[
= \tan^2 x + \frac{1}{\tan^2 x}
\]

\[
= \frac{1}{\cos^2 x}
\]

\[
= \frac{1}{1 - \sin^2 x}
\]

Since the left side is identical to the right side, the given equation is an identity.

Now try Exercise 37.
Example 3  Verifying an Identity (Working with One Side)

Verify that the following equation is an identity.

\[
\frac{\tan t - \cot t}{\sin t \cos t} = \sec^2 t - \csc^2 t
\]

Solution  We transform the more complicated left side to match the right side.

\[
\frac{\tan t - \cot t}{\sin t \cos t} = \frac{\tan t}{\sin t \cos t} - \frac{\cot t}{\sin t \cos t} = \frac{\frac{\sin t}{\cos t} - \frac{\cos t}{\sin t}}{\sin t \cos t} = \frac{\sin^2 t - \cos^2 t}{\sin t \cos t} \text{ (Section R.5)}
\]

\[
= \frac{\cos^2 t - \sin^2 t}{\sin t \cos t} \text{ (Section R.3)}
\]

\[
= \frac{\sec^2 t - \csc^2 t}{\tan t \csc t} \text{ (Section R.5)}
\]

The third hint about writing all trigonometric functions in terms of sine and cosine was used in the third line of the solution.

Now try Exercise 41.

Example 4  Verifying an Identity (Working with One Side)

Verify that the following equation is an identity.

\[
\frac{\cos x}{1 - \sin x} = \frac{1 + \sin x}{\cos x}
\]

Solution  We work on the right side, using the last hint in the list given earlier to multiply numerator and denominator on the right by \(1 + \sin x\).

\[
\frac{1 + \sin x}{\cos x} = \frac{(1 + \sin x)(1 - \sin x)}{\cos x(1 - \sin x)} \text{ (Multiply by 1. (Section R.1))}
\]

\[
= \frac{\cos x}{1 - \sin x} \text{ (\(x + y)(x - y) = x^2 - y^2\) (Section R.3)}
\]

\[
= \frac{\cos x}{1 - \sin x} \text{ (\(1 - \sin^2 x = \cos^2 x\))}
\]

\[
= \frac{\cos x}{1 - \sin x} \text{ (Lowest terms (Section R.5))}
\]

Now try Exercise 47.

Verifying Identities by Working with Both Sides  If both sides of an identity appear to be equally complex, the identity can be verified by working independently on the left side and on the right side, until each side is changed into some common third result. Each step, on each side, must be reversible.
With all steps reversible, the procedure is as shown in the margin. The left side leads to a common third expression, which leads back to the right side. This procedure is just a shortcut for the procedure used in Examples 1–4: one side is changed into the other side, but by going through an intermediate step.

**EXAMPLE 5 Verifying an Identity (Working with Both Sides)**

Verify that the following equation is an identity.

$$\frac{\sec \alpha + \tan \alpha}{\sec \alpha - \tan \alpha} = \frac{1 + 2 \sin \alpha + \sin^2 \alpha}{\cos^2 \alpha}$$

**Solution** Both sides appear equally complex, so we verify the identity by changing each side into a common third expression. We work first on the left, multiplying numerator and denominator by \(\cos \alpha\).

$$\frac{\sec \alpha + \tan \alpha}{\sec \alpha - \tan \alpha} = \frac{(\sec \alpha + \tan \alpha) \cos \alpha}{(\sec \alpha - \tan \alpha) \cos \alpha}$$

Multiply by 1.

$$= \frac{\sec \alpha \cos \alpha + \tan \alpha \cos \alpha}{\sec \alpha \cos \alpha - \tan \alpha \cos \alpha}$$

Distributive property

$$= \frac{1 + \tan \alpha \cos \alpha}{1 - \tan \alpha \cos \alpha}$$

\(\sec \alpha \cos \alpha = 1\)

$$= \frac{1 + \sin \alpha}{1 - \sin \alpha}$$

\(\tan \alpha = \frac{\sin \alpha}{\cos \alpha}\)

On the right side of the original equation, begin by factoring.

$$\frac{1 + 2 \sin \alpha + \sin^2 \alpha}{\cos^2 \alpha} = \frac{(1 + \sin \alpha)^2}{\cos^2 \alpha}$$

\(x^2 + 2xy + y^2 = (x+y)^2\)

(Section R.4)

$$= \frac{(1 + \sin \alpha)^2}{1 - \sin^2 \alpha}$$

\(\cos^2 \alpha = 1 - \sin^2 \alpha\)

$$= \frac{(1 + \sin \alpha)^2}{(1 + \sin \alpha)(1 - \sin \alpha)}$$

Factor \(1 - \sin^2 \alpha\).

$$= \frac{1 + \sin \alpha}{1 - \sin \alpha}$$

Lowest terms

We have shown that

$$\frac{\sec \alpha + \tan \alpha}{\sec \alpha - \tan \alpha} = \frac{1 + \sin \alpha}{1 - \sin \alpha} = \frac{1 + 2 \sin \alpha + \sin^2 \alpha}{\cos^2 \alpha},$$

verifying that the given equation is an identity.

Now try Exercise 51.
7.2 Verifying Trigonometric Identities

**CAUTION** Use the method of Example 5 only if the steps are reversible.

There are usually several ways to verify a given identity. For instance, another way to begin verifying the identity in Example 5 is to work on the left as follows.

\[
\frac{\sec \alpha + \tan \alpha}{\sec \alpha - \tan \alpha} = \frac{1}{\cos \alpha} + \frac{\sin \alpha}{\cos \alpha} \quad \text{Fundamental identities (Section 7.1)}
\]

\[
= \frac{1 + \sin \alpha}{\cos \alpha} \quad \text{Add and subtract fractions. (Section R.5)}
\]

\[
= \frac{1 + \sin \alpha}{1 - \sin \alpha} \quad \text{Simplify the complex fraction. (Section R.5)}
\]

Compare this with the result shown in Example 5 for the right side to see that the two sides indeed agree.

**EXAMPLE 6** Applying a Pythagorean Identity to Radios

Tuners in radios select a radio station by adjusting the frequency. A tuner may contain an inductor \(L\) and a capacitor \(C\), as illustrated in Figure 3. The energy stored in the inductor at time \(t\) is given by

\[L(t) = k \sin^2(2\pi Ft)\]

and the energy stored in the capacitor is given by

\[C(t) = k \cos^2(2\pi Ft),\]

where \(F\) is the frequency of the radio station and \(k\) is a constant. The total energy \(E\) in the circuit is given by

\[E(t) = L(t) + C(t).\]


**Solution**

\[E(t) = L(t) + C(t)\]

\[= k \sin^2(2\pi Ft) + k \cos^2(2\pi Ft)\]

\[= k[\sin^2(2\pi Ft) + \cos^2(2\pi Ft)]\]

\[= k(1)\]

\[= k\]

Since \(k\) is constant, \(E(t)\) is a constant function.

Now try Exercise 85.
7.2 Exercises

Perform each indicated operation and simplify the result.

1. \( \cot \theta + \frac{1}{\cot \theta} \)
2. \( \sec x + \frac{\csc x}{\sec x} \)
3. \( \tan s(\cot s + \csc s) \)
4. \( \cot \beta(\sec \beta + \csc \beta) \)
5. \( \frac{1}{\csc^2 \theta} + \frac{1}{\sec^2 \theta} \)
6. \( \frac{1}{\sin \alpha - 1} - \frac{1}{\sin \alpha + 1} \)
7. \( \frac{\cos x + \sin x}{\sec x} \)
8. \( \frac{\cot \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta} \)
9. \( (1 + \tan t)^2 + \cos^2 t \)
10. \( (1 + \tan s)^2 - 2 \tan s \)

Factor each trigonometric expression.

11. \( \frac{1}{\cos x - 1} - \frac{1}{1 - \cos x} \)
12. \( \sin^2 \alpha + \cos^2 \alpha \)

Each expression simplifies to a constant, a single function, or a power of a function. Use fundamental identities to simplify each expression.

13. \( \sin^2 \theta - 1 \)
14. \( \sec^2 \theta - 1 \)
15. \( (\sin x + 1)^2 - (\sin x - 1)^2 \)
16. \( (\tan x + \cot x)^2 - (\tan x - \cot x)^2 \)
17. \( 2 \sin^2 x + 3 \sin x + 1 \)
18. \( 4 \tan^2 \beta + \tan \beta - 3 \)
19. \( \cos^4 x + 2 \cos^2 x + 1 \)
20. \( \cot^4 x + 3 \cot^2 x + 2 \)
21. \( \sin^2 x - \cos^2 x \)
22. \( \sin^4 x + \sin x \csc x \)
23. \( \tan \theta \cos \theta \)
24. \( \cot \alpha \sin \alpha \)
25. \( \sec r \cos r \)
26. \( \cot t \tan t \)
27. \( \sin \beta \tan \beta \)
28. \( \csc \theta \sec \theta \)
29. \( \sec^2 x - 1 \)
30. \( \csc^2 t - 1 \)
31. \( \frac{\sin^2 x}{\cos^2 x} + \sin x \csc x \)
32. \( \frac{1}{\tan^2 \alpha} + \cot \alpha \tan \alpha \)

In Exercises 33–68, verify that each trigonometric equation is an identity. See Examples 1–5.

33. \( \cot \theta \csc \theta = \cos \theta \)
34. \( \tan \alpha = \sin \alpha \)
35. \( 1 - \sin^2 \beta = \cos \beta \)
36. \( \tan^2 \alpha + 1 = \sec \alpha \)
37. \( \cos \theta(\tan \theta + 1) = 1 \)
38. \( \sin^2 \beta(1 + \cot^2 \beta) = 1 \)
39. \( \cot s + \tan s = \sec s \csc s \)
40. \( \sin^2 \alpha + \tan^2 \alpha + \cos^2 \alpha = \sec^2 \alpha \)
41. \( \cos \alpha + \sin \alpha \csc \alpha = \sec^2 \alpha - \tan^2 \alpha \)
42. \( \sin^2 \theta \cos \theta = \sec \theta - \cos \theta \)
43. \( \sin^4 \theta - \cos^4 \theta = 2 \sin^2 \theta - 1 \)
44. \( \frac{\cos \theta}{\sin \theta \cot \theta} = 1 \)
45. \( (1 - \cos^2 \alpha)(1 + \cos^2 \alpha) = 2 \sin^2 \alpha - \sin^4 \alpha \)
46. \( \tan^2 \alpha \sin^2 \alpha = \tan^2 \alpha + \cos^2 \alpha - 1 \)
7.2 Verifying Trigonometric Identities

47. \( \frac{\cos \theta + 1}{\tan^2 \theta} = \frac{\cos \theta}{\sec \theta - 1} \)

48. \( \frac{(\sec \theta - \tan \theta)^2 + 1}{\sec \theta \csc \theta - \tan \theta \csc \theta} = 2 \tan \theta \)

49. \( \frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = 2 \sec^2 \theta \)

50. \( \frac{1}{\sec \alpha - \tan \alpha} = \sec \alpha + \tan \alpha \)

51. \( \frac{\tan s}{1 + \cos s} + \frac{\sin s}{1 - \cos s} = \cot s + \sec s \csc s \)

52. \( \frac{1 - \cos x}{1 + \cos x} = (\cot x - \csc x)^2 \)

53. \( \frac{\cot \alpha + 1}{\cot \alpha - 1} = 1 + \tan \alpha \)

54. \( \frac{1}{\tan \alpha - \sec \alpha} + \frac{1}{\tan \alpha + \sec \alpha} = -2 \tan \alpha \)

55. \( \sin^2 \alpha \sec^2 \alpha + \sin^2 \alpha \csc^2 \alpha = \sec^2 \alpha \)

56. \( \frac{\csc \theta + \cot \theta}{\tan \theta + \sin \theta} = \cot \theta \csc \theta \)

57. \( \sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x \)

58. \( \frac{1 - \sin \theta}{1 + \sin \theta} = \sec^2 \theta - 2 \sec \theta \tan \theta + \tan^2 \theta \)

59. \( \sin \theta + \cos \theta = \frac{\sin \theta}{1 - \sin \theta} + \frac{\cos \theta}{1 - \cos \theta} \)

60. \( \frac{\sin \theta}{1 - \cos \theta} - \frac{\sin \theta \cos \theta}{1 + \cos \theta} = \csc \theta (1 + \cos^2 \theta) \)

61. \( \frac{\sec^4 s - \tan^4 s}{\sec^2 s + \tan^2 s} = \sec^2 s - \tan^2 s \)

62. \( \frac{\cot^2 t - 1}{1 + \cot^2 t} = 1 - 2 \sin^2 t \)

63. \( \frac{\tan^2 t - 1}{\sec^2 t} = \frac{\tan t - \cot t}{\tan t + \cot t} \)

64. \( (1 + \sin x + \cos x)^2 = 2(1 + \sin x)(1 + \cos x) \)

65. \( \frac{1 + \cos x}{1 - \cos x} - \frac{1 - \cos x}{1 + \cos x} = 4 \cot x \csc x \)

66. \( (\sec \alpha - \tan \alpha)^2 = \frac{1 - \sin \alpha}{1 + \sin \alpha} \)

67. \( (\sec \alpha + \csc \alpha)(\cos \alpha - \sin \alpha) = \cot \alpha - \tan \alpha \)

68. \( \frac{\sin^2 \alpha - \cos^2 \alpha}{\sin^2 \alpha - \cos^2 \alpha} = 1 \)

69. A student claims that the equation \( \cos \theta + \sin \theta = 1 \) is an identity, since by letting \( \theta = 90^\circ \) (or \( \frac{\pi}{2} \) radians) we get \( 0 + 1 = 1 \), a true statement. Comment on this student’s reasoning.

70. An equation that is an identity has an infinite number of solutions. If an equation has an infinite number of solutions, is it necessarily an identity? Explain.
71. \((\sec \theta + \tan \theta)(1 - \sin \theta) = \cos \theta\)
72. \((\csc \theta + \cot \theta)(\sec \theta - 1) = \tan \theta\)
73. \(\frac{\cos \theta + 1}{\sin \theta + \tan \theta} = \cot \theta\)
74. \(\tan \theta \sin \theta + \cos \theta = \sec \theta\)
75. identity
76. identity
77. not an identity
78. not an identity
83. It is true when \(\sin x \geq 0\).
84. (a) \(I = k(1 - \sin^2 \theta)\)
(b) For \(\theta = 2\pi n\) for all integers \(n\), 
\(\cos^2 \theta = 1\), its maximum value, 
and \(I\) attains a maximum value of \(k\).
85. (a) \(P = 16k \cos^2(2\pi t)\)
(b) \(P = 16k[1 - \sin^2(2\pi t)]\)

**Concept Check**  Graph each expression and conjecture an identity. Then verify your 
conjecture algebraically.

71. \((\sec \theta + \tan \theta)(1 - \sin \theta) = \cos \theta\)
72. \((\csc \theta + \cot \theta)(\sec \theta - 1) = \tan \theta\)
73. \(\frac{\cos \theta + 1}{\sin \theta + \tan \theta} = \cot \theta\)
74. \(\tan \theta \sin \theta + \cos \theta = \sec \theta\)

Graph the expressions on each side of the equals sign to determine whether the equation 
might be an identity. (Note: Use a domain whose length is at least \(2\pi\).) If the equation 
looks like an identity, prove it algebraically. See Example 1.

75. \(\frac{2 + 5 \cos \theta}{\sin \theta} = 2 \csc \theta + 5 \cot \theta\)
76. \(1 + \cot^2 \theta = \frac{\sec^2 \theta}{\sec^2 \theta - 1}\)
77. \(\tan \theta - \cot \theta = 2 \sin^2 \theta\)
78. \(\frac{1 + \sin \theta}{1 - \sin \theta} = \sec^2 \theta\)

By substituting a number for \(s\) or \(t\), show that the equation is not an identity.

79. \(\sin(\csc \theta) = 1\)
80. \(\sqrt{\cos^2 \theta} = \cos \theta\)
81. \(\csc \theta = \sqrt{1 + \cot^2 \theta}\)
82. \(\cos \theta = \sqrt{1 - \sin^2 \theta}\)
83. When is \(\sin x = \sqrt{1 - \cos^2 x}\) a true statement?

**Modeling**  Work each problem.

84. **Intensity of a Lamp**  According to Lambert’s law, the 
intensity of light from a single source on a flat surface at 
point \(P\) is given by

\[ I = k \cos^2 \theta, \]


(a) Write \(I\) in terms of the sine function.
(b) Explain why the maximum value of \(I\) occurs when 
\(\theta = 0\).

85. **Oscillating Spring**  The distance or displacement \(y\) of a 
weight attached to an oscillating spring from its natural position is modeled by

\[ y = 4 \cos(2\pi t), \]

where \(t\) is time in seconds. Potential energy is the energy of 
position and is given by

\[ P = ky^2, \]

where \(k\) is a constant. The weight has the greatest potential energy when the spring 
is stretched the most. (Source: Weidner, R. and R. Sells, *Elementary Classical 

(a) Write an expression for \(P\) that involves the cosine function.
(b) Use a fundamental identity to write \(P\) in terms of \(\sin(2\pi t)\).

86. **Radio Tuners**  Refer to Example 6. Let the energy stored in the inductor be given by

\[ L(t) = 3 \cos^2(6,000,000t) \]

and the energy in the capacitor be given by

\[ C(t) = 3 \sin^2(6,000,000t), \]
where $t$ is time in seconds. The total energy $E$ in the circuit is given by $E(t) = L(t) + C(t)$.

(a) Graph $L$, $C$, and $E$ in the window $[0, 10^{-6}]$ by $[−1, 4]$, with $Xscl = 10^{-7}$ and $Yscl = 1$. Interpret the graph.

(b) Make a table of values for $L$, $C$, and $E$ starting at $t = 0$, incrementing by $10^{-7}$. Interpret your results.

(c) Use a fundamental identity to derive a simplified expression for $E(t)$.

$$E(t) = \frac{7.3}{t}$$

### 7.3 Sum and Difference Identities

**Cosine Sum and Difference Identities** Several examples presented earlier should have convinced you by now that $\cos(A - B) \text{ does not equal } \cos A - \cos B$. For example, if $A = \frac{\pi}{2}$ and $B = 0$, then

$$\cos(A - B) = \cos \left( \frac{\pi}{2} - 0 \right) = \cos \frac{\pi}{2} = 0,$$

while

$$\cos A - \cos B = \cos \frac{\pi}{2} - \cos 0 = 0 - 1 = -1.$$

We can now derive a formula for $\cos(A - B)$. We start by locating angles $A$ and $B$ in standard position on a unit circle, with $B < A$. Let $S$ and $Q$ be the points where the terminal sides of angles $A$ and $B$, respectively, intersect the circle. Locate point $R$ on the unit circle so that angle $POR$ equals the difference $A - B$. See Figure 4.

![Figure 4](image)

Point $Q$ is on the unit circle, so by the work with circular functions in Chapter 6, the $x$-coordinate of $Q$ is the cosine of angle $B$, while the $y$-coordinate of $Q$ is the sine of angle $B$.

$Q$ has coordinates $(\cos B, \sin B)$.

In the same way,

$S$ has coordinates $(\cos A, \sin A),$

and

$R$ has coordinates $(\cos(A - B), \sin(A - B))$. 

### Example

(b) Let $Y_1 = L(t)$, $Y_2 = C(t)$, and $Y_3 = E(t)$. $Y_3 = 3$ for all inputs.

(e) $E(t) = 3$
Angle $SOQ$ also equals $A - B$. Since the central angles $SOQ$ and $POR$ are equal, chords $PR$ and $SQ$ are equal. By the distance formula, since $PR = SQ$,

$$\sqrt{[\cos(A - B) - 1]^2 + [\sin(A - B) - 0]^2} = \sqrt{\cos^2 A - \cos B + \sin A - \sin B}. \quad (\text{Section 2.1})$$

Squaring both sides and clearing parentheses gives

$$\cos^2(A - B) - 2 \cos(A - B) + 1 + \sin^2(A - B) = \cos^2 A - 2 \cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B.$$ 

Since $\sin^2 x + \cos^2 x = 1$ for any value of $x$, we can rewrite the equation as

$$2 - 2 \cos(A - B) = 2 - 2 \cos A \cos B - 2 \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \quad \text{Subtract 2; divide by -2.}$$

Although Figure 4 shows angles $A$ and $B$ in the second and first quadrants, respectively, this result is the same for any values of these angles.

To find a similar expression for $\cos(A + B)$, rewrite $A + B$ as $A - (-B)$ and use the identity for $\cos(A - B)$.

$$\cos(A + B) = \cos[A - (-B)]$$

$$= \cos A \cos(-B) + \sin A \sin(-B) \quad \text{Cosine difference identity}$$

$$= \cos A \cos B + \sin A(-\sin B) \quad \text{Negative-angle identities (Section 7.1)}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

### Cosine of a Sum or Difference

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

These identities are important in calculus and useful in certain applications. Although a calculator can be used to find an approximation for $\cos 15^\circ$, for example, the method shown below can be applied to get an exact value, as well as to practice using the sum and difference identities.

**EXAMPLE 1 Finding Exact Cosine Function Values**

Find the exact value of each expression.

(a) $\cos 15^\circ$  
(b) $\cos \frac{5\pi}{12}$  
(c) $\cos 87^\circ \cos 93^\circ - \sin 87^\circ \sin 93^\circ$

**Solution**

(a) To find $\cos 15^\circ$, we write $15^\circ$ as the sum or difference of two angles with known function values. Since we know the exact trigonometric function values of $45^\circ$ and $30^\circ$, we write $15^\circ$ as $45^\circ - 30^\circ$. (We could also use $60^\circ - 45^\circ$.) Then we use the identity for the cosine of the difference of two angles.
Teaching Tip In Example 1(b), students may benefit from converting \( \frac{5\pi}{12} \) radians to 75° in order to realize that
\[
\frac{\pi}{6} + \frac{\pi}{4} = 30° + 45°
\]
can be used in place of \( \frac{5\pi}{12} \).

The screen supports the solution in Example 1(b) by showing that
\[
\cos \left( \frac{5\pi}{12} \right) = \frac{\sqrt{6} - \sqrt{2}}{4}
\]

Now try Exercises 7, 9, and 11.

Cofunction Identities We can use the identity for the cosine of the difference of two angles and the fundamental identities to derive cofunction identities.

Cofunction Identities

\[
\begin{align*}
\cos(90° - \theta) & = \sin \theta \\
\cot(90° - \theta) & = \tan \theta \\
\sin(90° - \theta) & = \cos \theta \\
\sec(90° - \theta) & = \csc \theta \\
\tan(90° - \theta) & = \cot \theta \\
\csc(90° - \theta) & = \sec \theta
\end{align*}
\]

Similar identities can be obtained for a real number domain by replacing 90° with \( \frac{\pi}{2} \).

Substituting 90° for \( A \) and \( \theta \) for \( B \) in the identity for \( \cos(A - B) \) gives
\[
\begin{align*}
\cos(90° - \theta) & = \cos 90° \cos \theta + \sin 90° \sin \theta \\
& = 0 \cdot \cos \theta + 1 \cdot \sin \theta \\
& = \sin \theta.
\end{align*}
\]

This result is true for any value of \( \theta \) since the identity for \( \cos(A - B) \) is true for any values of \( A \) and \( B \).
EXAMPLE 2 Using Cofunction Identities to Find $\theta$

Find an angle $\theta$ that satisfies each of the following.

(a) $\cot \theta = \tan 25^\circ$
(b) $\sin \theta = \cos(-30^\circ)$
(c) $\csc \frac{3\pi}{4} = \sec \theta$

Solution

(a) Since tangent and cotangent are cofunctions, $\tan(90^\circ - \theta) = \cot \theta$.

\[
\begin{align*}
\cot \theta &= \tan 25^\circ \\
\tan(90^\circ - \theta) &= \tan 25^\circ & \text{Cofunction identity} \\
90^\circ - \theta &= 25^\circ & \text{Set angle measures equal.} \\
\theta &= 65^\circ
\end{align*}
\]

(b) $\sin \theta = \cos(-30^\circ)$

\[
\begin{align*}
\cos(90^\circ - \theta) &= \cos(-30^\circ) & \text{Cofunction identity} \\
90^\circ - \theta &= -30^\circ \\
\theta &= 120^\circ
\end{align*}
\]

(c) $\csc \frac{3\pi}{4} = \sec \theta$

\[
\begin{align*}
\sec\left(\frac{\pi}{2} - \frac{3\pi}{4}\right) &= \sec \theta & \text{Cofunction identity} \\
\sec\left(-\frac{\pi}{4}\right) &= \sec \theta & \text{Combine terms.} \\
-\frac{\pi}{4} &= \theta
\end{align*}
\]

Now try Exercises 25 and 27.

Note Because trigonometric (circular) functions are periodic, the solutions in Example 2 are not unique. We give only one of infinitely many possibilities.

If one of the angles $A$ or $B$ in the identities for $\cos(A + B)$ and $\cos(A - B)$ is a quadrantal angle, then the identity allows us to write the expression in terms of a single function of $A$ or $B$.

EXAMPLE 3 Reducing $\cos(A - B)$ to a Function of a Single Variable

Write $\cos(180^\circ - \theta)$ as a trigonometric function of $\theta$.

Solution Use the difference identity. Replace $A$ with $180^\circ$ and $B$ with $\theta$.

\[
\begin{align*}
\cos(180^\circ - \theta) &= \cos 180^\circ \cos \theta + \sin 180^\circ \sin \theta \\
&= (-1) \cos \theta + (0) \sin \theta & \text{(Section 5.2)} \\
&= -\cos \theta
\end{align*}
\]

Now try Exercise 39.
7.3 Sum and Difference Identities 625

**Sine and Tangent Sum and Difference Identities** We can use the cosine sum and difference identities to derive similar identities for sine and tangent. Since \( \sin \theta = \cos(90^\circ - \theta) \), we replace \( \theta \) with \( A + B \) to get

\[
\sin(A + B) = \cos(90^\circ - (A + B)) \quad \text{Cofunction identity}
\]
\[
= \cos(90^\circ - A - B) \quad \text{Cosine difference identity}
\]
\[
= \cos(90^\circ - A) \cos B + \sin(90^\circ - A) \sin B
\]

Now we write \( \sin(A - B) \) as \( \sin[A + (-B)] \) and use the identity for \( \sin(A + B) \).

\[
\sin(A - B) = \sin[A + (-B)]
\]
\[
= \sin A \cos(-B) + \cos A \sin(-B) \quad \text{Sine sum identity}
\]
\[
= \sin A \cos B - \cos A \sin B \quad \text{Negative-angle identities}
\]

**Sine of a Sum or Difference**

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]
\[
\sin(A - B) = \sin A \cos B - \cos A \sin B
\]

To derive the identity for \( \tan(A + B) \), we start with

\[
\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} \quad \text{Fundamental identity (Section 7.1)}
\]
\[
= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \quad \text{Sum identities}
\]

We express this result in terms of the tangent function by multiplying both numerator and denominator by \( \frac{1}{\cos A \cos B} \).

\[
\tan(A + B) = \frac{1}{\cos A \cos B - \sin A \sin B} \cdot \frac{1}{\cos A \cos B}
\]
\[
= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B} \quad \text{Simplify the complex fraction. (Section R.5)}
\]
\[
= \frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}
\]
\[
= \frac{\cos A \cos B}{\cos A \cos B} \quad \text{Multiply numerators; multiply denominators.}
\]
\[
= \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}
\]
\[
= \frac{1}{1 - \frac{\sin A \sin B}{\cos A \cos B}} \quad \text{Fundamental identity (Section 7.1)}
\]
\[
= \frac{1}{1 - \tan A \tan B}
\]

Replacing \( B \) with \( -B \) and using the fact that \( \tan(-B) = -\tan B \) gives the identity for the tangent of the difference of two angles.
 EXAMPLE 4 Finding Exact Sine and Tangent Function Values

Find the exact value of each expression.

(a) \( \sin 75^\circ \)  
(b) \( \tan \frac{7\pi}{12} \)  
(c) \( \sin 40^\circ \cos 160^\circ - \cos 40^\circ \sin 160^\circ \)

**Solution**

(a) \( \sin 75^\circ = \sin(45^\circ + 30^\circ) \)
   \[ = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \quad \text{Sine sum identity} \]
   \[ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \quad \text{(Section 5.3)} \]
   \[ = \frac{\sqrt{6} + \sqrt{2}}{4} \]

(b) \( \tan \frac{7\pi}{12} = \tan \left( \frac{\pi}{3} + \frac{\pi}{4} \right) \)
   \[ = \frac{\tan \frac{\pi}{3} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{3} \tan \frac{\pi}{4}} \quad \text{Tangent sum identity} \]
   \[ = \frac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1} \quad \text{(Section 6.2)} \]
   \[ = \frac{\sqrt{3} + 1}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \quad \text{Rationalize the denominator. (Section R.7)} \]
   \[ = \frac{\sqrt{3} + 3 + 1 + \sqrt{3}}{1 - 3} \quad \text{Multiply. (Section R.7)} \]
   \[ = \frac{4 + 2\sqrt{3}}{-2} \quad \text{Combine terms.} \]
   \[ = \frac{2(2 + \sqrt{3})}{-2} \quad \text{Factor out 2. (Section R.5)} \]
   \[ = -2 - \sqrt{3} \quad \text{Lowest terms} \]

(c) \( \sin 40^\circ \cos 160^\circ - \cos 40^\circ \sin 160^\circ = \sin(40^\circ - 160^\circ) \)
   \[ = \sin(-120^\circ) \]
   \[ = -\sin 120^\circ \quad \text{Negative-angle identity} \]
   \[ = -\frac{\sqrt{3}}{2} \quad \text{(Section 5.3)} \]

Now try Exercises 29, 31, and 35.
EXAMPLE 5 Writing Functions as Expressions Involving Functions of \( \theta \)

Write each function as an expression involving functions of \( \theta \).

(a) \( \sin(30^\circ + \theta) \)  (b) \( \tan(45^\circ - \theta) \)  (c) \( \sin(180^\circ + \theta) \)

Solution

(a) Using the identity for \( \sin(A + B) \),
\[
\sin(30^\circ + \theta) = \sin 30^\circ \cos \theta + \cos 30^\circ \sin \theta
\]
\[
= \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta.
\]

(b) \( \tan(45^\circ - \theta) = \frac{\tan 45^\circ - \tan \theta}{1 + \tan 45^\circ \tan \theta} = \frac{1 - \tan \theta}{1 + \tan \theta} \)

(c) \( \sin(180^\circ + \theta) = \sin 180^\circ \cos \theta + \cos 180^\circ \sin \theta \)
\[
= 0 \cdot \cos \theta + (-1) \sin \theta
\]
\[
= -\sin \theta
\]

Now try Exercises 43 and 47.

EXAMPLE 6 Finding Function Values and the Quadrant of \( A + B \)

Suppose that \( A \) and \( B \) are angles in standard position, with \( \sin A = \frac{4}{5} \), \( \frac{\pi}{2} < A < \pi \), and \( \cos B = -\frac{3}{5} \), \( \pi < B < \frac{3\pi}{2} \). Find each of the following.

(a) \( \sin(A + B) \)  (b) \( \tan(A + B) \)  (c) the quadrant of \( A + B \)

Solution

(a) The identity for \( \sin(A + B) \) requires \( \sin A \), \( \cos A \), \( \sin B \), and \( \cos B \). We are given values of \( \sin A \) and \( \cos B \). We must find values of \( \cos A \) and \( \sin B \).
\[
\sin^2 A + \cos^2 A = 1 \quad \text{Fundamental identity}
\]
\[
\frac{16}{25} + \cos^2 A = 1 \quad \sin A = \frac{4}{5}
\]
\[
\cos^2 A = \frac{9}{25} \quad \text{Subtract} \frac{16}{25}.
\]
\[
\cos A = -\frac{3}{5} \quad \text{Since} A \text{ is in quadrant II, } \cos A < 0.
\]

In the same way, \( \sin B = -\frac{12}{13} \). Now use the formula for \( \sin(A + B) \).
\[
\sin(A + B) = \frac{4}{5} \left( -\frac{5}{13} \right) + \left( -\frac{3}{5} \right) \left( -\frac{12}{13} \right)
\]
\[
= -\frac{20}{65} + \frac{36}{65} = \frac{16}{65}
\]

(b) To find \( \tan(A + B) \), first use the values of sine and cosine from part (a) to get \( \tan A = -\frac{4}{3} \) and \( \tan B = \frac{12}{5} \).
\[
\tan(A + B) = \frac{-\frac{4}{3} + \frac{12}{5}}{1 - \left( -\frac{4}{3} \right) \left( \frac{12}{5} \right)} = \frac{16}{15} + \frac{63}{15} = \frac{16}{15} = \frac{16}{63}
\]
(c) From parts (a) and (b), \( \sin(A + B) = \frac{16}{63} \) and \( \tan(A + B) = \frac{16}{63} \), both positive. Therefore, \( A + B \) must be in quadrant I, since it is the only quadrant in which both sine and tangent are positive.

Now try Exercise 51.

**EXAMPLE 7** Applying the Cosine Difference Identity to Voltage

Common household electric current is called *alternating current* because the current alternates direction within the wires. The voltage \( V \) in a typical 115-volt outlet can be expressed by the function \( V(t) = 163 \sin \omega t \), where \( \omega \) is the angular speed (in radians per second) of the rotating generator at the electrical plant and \( t \) is time measured in seconds. (*Source: Bell, D., Fundamentals of Electric Circuits, Fourth Edition, Prentice-Hall, 1988.*)

(a) It is essential for electric generators to rotate at precisely 60 cycles per sec so household appliances and computers will function properly. Determine \( \omega \) for these electric generators.

(b) Graph \( V \) in the window \([0, .05]\) by \([-200, 200]\).

(c) Determine a value of \( \phi \) so that the graph of \( V(t) = 163 \cos(\omega t - \phi) \) is the same as the graph of \( V(t) = 163 \sin \omega t \).

**Solution**

(a) Each cycle is \( 2\pi \) radians at 60 cycles per sec, so the angular speed is \( \omega = 60(2\pi) = 120\pi \) radians per sec.

(b) \( V(t) = 163 \sin \omega t = 163 \sin 120\pi t \). Because the amplitude of the function is 163 (from Section 6.3), \([-200, 200]\) is an appropriate interval for the range, as shown in Figure 5.

For \( x = t \),
\[
V(t) = 163 \sin 120\pi t
\]

![Figure 5](image)

(c) Using the negative-angle identity for cosine and a cofunction identity,
\[
\cos\left(x - \frac{\pi}{2}\right) = \cos\left[-\left(\frac{\pi}{2} - x\right)\right] = \cos\left(\frac{\pi}{2} - x\right) = \sin x.
\]

Therefore, if \( \phi = \frac{\pi}{2} \), then
\[
V(t) = 163 \cos\left(\omega t - \frac{\pi}{2}\right) = 163 \sin \omega t.
\]

Now try Exercise 81.
7.3 Exercises

11. 5. 6. 7. 8. 9. 10. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38.

7.3 Exercises

11. 5. 6. 7. 8. 9. 10. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38.

Concept Check  Match each expression in Column I with the correct expression in Column II to form an identity.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \cos(x + y) = )</td>
<td>A. ( \cos x \cos y + \sin x \sin y )</td>
</tr>
<tr>
<td>2. ( \cos(x - y) = )</td>
<td>B. ( \sin x \sin y - \cos x \cos y )</td>
</tr>
<tr>
<td>3. ( \sin(x + y) = )</td>
<td>C. ( \sin x \cos y + \cos x \sin y )</td>
</tr>
<tr>
<td>4. ( \sin(x - y) = )</td>
<td>D. ( \sin x \cos y - \cos x \sin y )</td>
</tr>
<tr>
<td>E. ( \cos x \sin y - \sin x \cos y )</td>
<td></td>
</tr>
<tr>
<td>F. ( \cos x \cos y - \sin x \sin y )</td>
<td></td>
</tr>
</tbody>
</table>

Use identities to find each exact value. (Do not use a calculator.) See Example 1.

5. \( \cos 75^\circ \)
6. \( \cos(-15^\circ) \)
7. \( \cos 105^\circ \)
8. \( \cos(-105^\circ) \)
(Hint: \( 105^\circ = 60^\circ + 45^\circ \))
9. \( \cos \left( \frac{7\pi}{12} \right) \)
10. \( \cos \left( -\frac{\pi}{12} \right) \)
11. \( \cos 40^\circ \cos 50^\circ - \sin 40^\circ \sin 50^\circ \)
12. \( \cos \frac{7\pi}{9} \cos \frac{2\pi}{9} - \sin \frac{7\pi}{9} \sin \frac{2\pi}{9} \)

Write each function value in terms of the cofunction of a complementary angle. See Example 2.

13. \( \tan 87^\circ \)
14. \( \sin 15^\circ \)
15. \( \cos \frac{\pi}{12} \)
16. \( \sin \frac{2\pi}{5} \)
17. \( \sin \frac{5\pi}{8} \)
18. \( \cot \frac{9\pi}{10} \)
19. \( \sec 146^\circ 42' \)
20. \( \tan 174^\circ 3' \)

Use the cofunction identities to fill in each blank with the appropriate trigonometric function name. See Example 2.

21. \( \cot \frac{\pi}{3} = \) \( \frac{\pi}{6} \)
22. \( \sin \frac{2\pi}{3} = \) \( -\frac{\pi}{6} \)
23. \( 33^\circ = \sin 57^\circ \)
24. \( 72^\circ = \cot 18^\circ \)

Find an angle \( \theta \) that makes each statement true. See Example 2.

25. \( \tan \theta = \cot(45^\circ + 2\theta) \)
26. \( \sin \theta = \cos(2\theta - 10^\circ) \)
27. \( \sin(3\theta - 15^\circ) = \cos(\theta + 25^\circ) \)
28. \( \cot(\theta - 10^\circ) = \tan(2\theta + 20^\circ) \)

Use identities to find the exact value of each of the following. See Example 4.

29. \( \sin \frac{5\pi}{12} \)
30. \( \tan \frac{5\pi}{12} \)
31. \( \tan \frac{\pi}{12} \)
32. \( \sin \frac{\pi}{12} \)
33. \( \sin \left( -\frac{7\pi}{12} \right) \)
34. \( \tan \left( -\frac{7\pi}{12} \right) \)
35. \( \sin 76^\circ \cos 31^\circ - \cos 76^\circ \sin 31^\circ \)
36. \( \sin 40^\circ \cos 50^\circ + \cos 40^\circ \sin 50^\circ \)
37. \( \tan 80^\circ + \tan 55^\circ \)
38. \( \tan 80^\circ - \tan(55^\circ) \)
39. \( 1 - \tan 80^\circ \tan 55^\circ \)
40. \( 1 + \tan 80^\circ \tan(-55^\circ) \)
### 39. sin \( \theta \) 40. \(-\cos \theta\) 41. \(-\sin x\) 42. \(-\sin x\)  
43. \(\frac{\sqrt{3}}{2}(\cos \theta + \sin \theta)\) 44. sin \( \theta \) 45. \(\frac{1}{2}(\cos x - \sqrt{3}\sin x)\) 46. \(\frac{\sqrt{2}}{2}(\cos x - \sin x)\) 47. \(\frac{3}{2} + \tan \theta\) 48. \(\frac{\sqrt{3} \tan \theta - 1}{\sqrt{3} + \tan \theta}\) 49. \(\frac{\sqrt{3} \tan x + 1}{\sqrt{3} - \tan x}\) 50. \(\frac{1 + \tan x}{1 - \tan x}\)

### Use identities to write each expression as a function of \( x \) or \( \theta \). See Examples 3 and 5.

39. \(\cos(90^\circ - \theta)\) 40. \(\cos(180^\circ - \theta)\) 41. \(\cos\left(x - \frac{3\pi}{2}\right)\) 42. \(\cos\left(\frac{\pi}{2} + x\right)\) 43. \(\sin(45^\circ + \theta)\) 44. \(\sin(180^\circ - \theta)\) 45. \(\sin\left(\frac{5\pi}{6} - x\right)\) 46. \(\sin\left(\frac{\pi}{4} - x\right)\) 47. \(\tan(60^\circ + \theta)\) 48. \(\tan(\theta - 30^\circ)\) 49. \(\tan\left(x + \frac{\pi}{6}\right)\) 50. \(\tan\left(\frac{\pi}{4} + x\right)\)

### Use the given information to find (a) \(\cos(s + t)\), (b) \(\sin(s - t)\), (c) \(\tan(s + t)\), and (d) the quadrant of \( s + t \). See Example 6.

51. \(\cos s = \frac{3}{5}\) and \(\sin t = \frac{5}{13}\), \( s \) and \( t \) in quadrant I 52. \(\cos s = -\frac{1}{5}\) and \(\sin t = \frac{3}{5}\), \( s \) and \( t \) in quadrant II 53. \(\sin s = \frac{2}{3}\) and \(\sin t = -\frac{1}{3}\), \( s \) in quadrant II and \( t \) in quadrant IV 54. \(\sin s = \frac{3}{5}\) and \(\sin t = -\frac{12}{13}\), \( s \) in quadrant I and \( t \) in quadrant III 55. \(\cos s = -\frac{8}{17}\) and \(\cos t = -\frac{3}{5}\), \( s \) and \( t \) in quadrant III 56. \(\cos s = -\frac{15}{17}\) and \(\sin t = \frac{4}{5}\), \( s \) in quadrant II and \( t \) in quadrant I

### Relating Concepts

*For individual or collaborative investigation (Exercises 57–60)*

*The identities for \(\cos(A + B)\) and \(\cos(A - B)\) can be used to find exact values of expressions like \(\cos 195^\circ\) and \(\cos 255^\circ\), where the angle is not in the first quadrant. Work Exercises 57–60 in order, to see how this is done.*

57. By writing \(195^\circ\) as \(180^\circ + 15^\circ\), use the identity for \(\cos(A + B)\) to express \(\cos 195^\circ\) as \(-\cos 15^\circ\). 58. Use the identity for \(\cos(A - B)\) to find \(-\cos 15^\circ\). 59. By the results of Exercises 57 and 58, \(\cos 195^\circ = \boxed{}\). 60. Find each exact value using the method shown in Exercises 57–59.

(a) \(\cos 255^\circ\) \hspace{1cm} (b) \(\cos \frac{11\pi}{12}\)

### Find each exact value. Use the technique developed in Relating Concepts Exercises 57–60.

61. \(\sin 165^\circ\) \hspace{1cm} 62. \(\tan 165^\circ\) \hspace{1cm} 63. \(\sin 255^\circ\) 64. \(\tan 285^\circ\) \hspace{1cm} 65. \(\tan \frac{11\pi}{12}\) \hspace{1cm} 66. \(\sin \left(-\frac{13\pi}{12}\right)\)
7.3 Sum and Difference Identities 631

71. \( \sin \left( \frac{\pi}{2} + x \right) = \cos x \)

72. \( \frac{1 + \tan x}{1 - \tan x} = \tan \left( \frac{\pi}{4} + x \right) \)

79. 3 80. 163 and -163; no

81. (a) 425 lb (c) 0°

67. Use the identity \( \cos(90^\circ - \theta) = \sin \theta \), and replace \( \theta \) with \( 90^\circ - A \), to derive the identity \( \cos A = \sin(90^\circ - A) \).

68. Explain how the identities for \( \sec(A + B) \), \( \csc(A + B) \), and \( \cot(A + B) \) can be found by using the sum identities given in this section.

69. Why is it not possible to use a method similar to that of Example 5(c) to find a formula for \( \tan(270^\circ - \theta) \)?

70. Concept Check  Show that if \( A \), \( B \), and \( C \) are angles of a triangle, then

\[
\sin(A + B + C) = 0.
\]

Graph each expression and use the graph to conjecture an identity. Then verify your conjecture algebraically.

71. \( \sin \left( \frac{\pi}{2} + x \right) \)

72. \( \frac{1 + \tan x}{1 - \tan x} \)

Verify that each equation is an identity.

73. \( \sin(x + y) + \sin(x - y) = 2 \sin x \cos y \)

74. \( \tan(x - y) - \tan(y - x) = \frac{2(\tan x - \tan y)}{1 + \tan x \tan y} \)

75. \( \frac{\cos(\alpha - \beta)}{\cos \alpha \sin \beta} = \tan \alpha + \cot \beta \)

76. \( \frac{\sin(s + t)}{\cos s \cos t} = \tan s + \tan t \)

77. \( \frac{\sin(x - y)}{\cos y} = \frac{\tan x - \tan y}{\tan x + \tan y} \)

78. \( \frac{\sin(s - t)}{\sin t} + \frac{\cos(s - t)}{\cos t} = \frac{\sin s}{\sin t \cos t} \)

Exercises 79 and 80 refer to Example 7.

79. How many times does the current oscillate in .05 sec?

80. What are the maximum and minimum voltages in this outlet? Is the voltage always equal to 115 volts?

(Modeling) Solve each problem.

81. Back Stress  If a person bends at the waist with a straight back making an angle of \( \theta \) degrees with the horizontal, then the force \( F \) exerted on the back muscles can be modeled by the equation

\[
F = \frac{.6W \sin(\theta + 90^\circ)}{\sin 12^\circ},
\]

where \( W \) is the weight of the person. (Source: Metcalf, H.,Topics in Classical Biophysics, Prentice-Hall, 1980.)

(a) Calculate \( F \) when \( W = 170 \text{ lb} \) and \( \theta = 30^\circ \).

(b) Use an identity to show that \( F \) is approximately equal to \( 2.9W \cos \theta \).

(c) For what value of \( \theta \) is \( F \) maximum?
82. (a) 408 lb (b) 46.1°
83. (a) The pressure $P$ is oscillating.

For $x = t$,
$$P(t) = \frac{4}{10} \cos \left[ \frac{20\pi}{4.9} - 1026t \right]$$

(b) The pressure oscillates and amplitude decreases as $r$ increases.

For $x = r$,
$$P(r) = \frac{3}{2} \cos \left[ \frac{2\pi r}{4.9} - 10,260 \right]$$

(c) $P = \frac{a}{n\lambda} \cos ct$

84. (a) For $x = t$,
$$V = V_1 + V_2 = 30 \sin 120\pi t + 40 \cos 120\pi t$$

(b) $a = 50; \phi = -5.353$

82. **Back Stress** Refer to Exercise 81.

(a) Suppose a 200-lb person bends at the waist so that $\theta = 45^\circ$. Estimate the force exerted on the person’s back muscles.

(b) Approximate graphically the value of $\theta$ that results in the back muscles exerting a force of 400 lb.

83. **Sound Waves** Sound is a result of waves applying pressure to a person’s eardrum. For a pure sound wave radiating outward in a spherical shape, the trigonometric function defined by

$$P = \frac{a}{r} \cos \left( \frac{2\pi r}{\lambda} - ct \right)$$

can be used to model the sound pressure at a radius of $r$ feet from the source, where $t$ is time in seconds, $\lambda$ is length of the sound wave in feet, $c$ is speed of sound in feet per second, and $a$ is maximum sound pressure at the source measured in pounds per square foot. (*Source: Beranek, L., Noise and Vibration Control, Institute of Noise Control Engineering, Washington, D.C., 1988.*) Let $\lambda = 4.9$ ft and $c = 1026$ ft per sec.

(a) Let $a = .4$ lb per ft$^2$. Graph the sound pressure at distance $r = 10$ ft from its source in the window $[0.05] \times [0.05]$. Describe $P$ at this distance.

(b) Now let $a = 3$ and $t = 10$. Graph the sound pressure in the window $[0, 20]$ by $[0, .05]$. What happens to pressure $P$ as radius $r$ increases?

(c) Suppose a person stands at a radius $r$ so that $r = n\lambda$, where $n$ is a positive integer. Use the difference identity for cosine to simplify $P$ in this situation.

84. **Voltage of a Circuit** When the two voltages

$$V_1 = 30 \sin 120\pi t \quad \text{and} \quad V_2 = 40 \cos 120\pi t$$

are applied to the same circuit, the resulting voltage $V$ will be equal to their sum. (*Source: Bell, D., Fundamentals of Electric Circuits, Second Edition, Reston Publishing Company, 1981.*)

(a) Graph the sum in the window $[0.05] \times [-60, 60]$.

(b) Use the graph to estimate values for $a$ and $\phi$ so that $V = a \sin(120\pi t + \phi)$.

(c) Use identities to verify that your expression for $V$ is valid.

### 7.4 Double-Angle Identities and Half-Angle Identities

#### Double-Angle Identities

When $A = B$ in the identities for the sum of two angles, these identities are called the double-angle identities. For example, to derive an expression for $\cos 2A$, we let $B = A$ in the identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

$$\cos 2A = \cos(A + A) = \cos A \cos A - \sin A \sin A \quad \text{Cosine sum identity (Section 7.3)}$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
Two other useful forms of this identity can be obtained by substituting either \( \cos^2 A = 1 - \sin^2 A \) or \( \sin^2 A = 1 - \cos^2 A \). Replace \( \cos^2 A \) with the expression \( 1 - \sin^2 A \) to get

\[
\cos 2A = \cos^2 A - \sin^2 A
= (1 - \sin^2 A) - \sin^2 A \quad \text{(Fundamental identity (Section 7.1))}
\]

\[
\cos 2A = 1 - 2 \sin^2 A,
\]

or replace \( \sin^2 A \) with \( 1 - \cos^2 A \) to get

\[
\cos 2A = \cos^2 A - \sin^2 A
= \cos^2 A - (1 - \cos^2 A) \quad \text{(Fundamental identity)}
= \cos^2 A - 1 + \cos^2 A
\cos 2A = 2 \cos^2 A - 1.
\]

We find \( \sin 2A \) with the identity \( \sin(A + B) = \sin A \cos B + \cos A \sin B \), letting \( B = A \).

\[
\sin 2A = \sin(A + A)
= \sin A \cos A + \cos A \sin A \quad \text{(Sine sum identity)}
\sin 2A = 2 \sin A \cos A
\]

Using the identity for \( \tan(A + B) \), we find \( \tan 2A \).

\[
\tan 2A = \tan(A + A)
= \frac{\tan A + \tan A}{1 - \tan A \tan A} \quad \text{(Tangent sum identity)}
\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}
\]

### Double-Angle Identities

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos 2A = \cos^2 A - \sin^2 A )</td>
<td>( \cos 2A = 1 - 2 \sin^2 A )</td>
</tr>
<tr>
<td>( \cos 2A = 2 \cos^2 A - 1 )</td>
<td>( \sin 2A = 2 \sin A \cos A )</td>
</tr>
<tr>
<td>( \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} )</td>
<td></td>
</tr>
</tbody>
</table>

### Example 1 Finding Function Values of 2\( \theta \) Given Information about \( \theta \)

Given \( \cos \theta = \frac{3}{5} \) and \( \sin \theta < 0 \), find \( \sin 2\theta \), \( \cos 2\theta \), and \( \tan 2\theta \).

**Solution** To find \( \sin 2\theta \), we must first find the value of \( \sin \theta \).

\[
\sin^2 \theta + \left( \frac{3}{5} \right)^2 = 1 \quad \sin^2 \theta + \cos^2 \theta = 1; \cos \theta = \frac{3}{5}
\]

\[
\sin^2 \theta = \frac{16}{25}
\]

\[
\sin \theta = -\frac{4}{5} \quad \text{Simplify.}
\]

\[
\sin \theta = -\frac{4}{5} \quad \text{Choose the negative square root since \( \sin \theta < 0 \).}
\]
Using the double-angle identity for sine,
\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]
\[
= 2 \left( -\frac{4}{5} \right) \left( \frac{3}{5} \right) = -\frac{24}{25}. \quad \sin \theta = -\frac{4}{5}; \cos \theta = \frac{3}{5}
\]
Now we find \( \cos 2\theta \), using the first of the double-angle identities for cosine. (Any of the three forms may be used.)
\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}
\]
The value of \( \tan 2\theta \) can be found in either of two ways. We can use the double-angle identity and the fact that \( \tan \theta = \frac{\sin \theta}{\cos \theta} = -\frac{4}{3} \)
\[
\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \left( -\frac{4}{3} \right)}{1 - \left( -\frac{4}{3} \right)^2} = \frac{24}{7}
\]
Simplify. (Section R.5)

Alternatively, we can find \( \tan 2\theta \) by finding the quotient of \( \sin 2\theta \) and \( \cos 2\theta \).
\[
\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7}
\]

**EXAMPLE 2  Verifying a Double-Angle Identity**

Verify that the following equation is an identity.
\[
\cot x \sin 2x = 1 + \cos 2x
\]

**Solution**  We start by working on the left side, using the hint from Section 7.1 about writing all functions in terms of sine and cosine.
\[
\cot x \sin 2x = \frac{\cos x}{\sin x} \cdot \sin 2x \quad \text{Quotient identity}
\]
\[
= \frac{\cos x}{\sin x} \left( 2 \sin x \cos x \right) \quad \text{Double-angle identity}
\]
\[
= 2 \cos^2 x
\]
\[
= 1 + \cos 2x \quad \text{cos } 2x = 2 \cos^2 x - 1, \text{ so}
\]
\[
2 \cos^2 x = 1 + \cos 2x
\]

The final step illustrates the importance of being able to recognize alternative forms of identities.
EXAMPLE 3  Simplifying Expressions Using Double-Angle Identities

Simplify each expression.

(a) \(\cos^2 7x - \sin^2 7x\)  
(b) \(\sin 15^\circ \cos 15^\circ\)

Solution

(a) This expression suggests one of the double-angle identities for cosine:
\[
\cos 2A = \cos^2 A - \sin^2 A.
\]
Substituting \(7x\) for \(A\) gives
\[
\cos^2 7x - \sin^2 7x = \cos 2(7x) = \cos 14x.
\]

(b) If this expression were \(2 \sin 15^\circ \cos 15^\circ\), we could apply the identity for \(\sin 2A\) directly since \(\sin 2A = 2 \sin A \cos A\). We can still apply the identity with \(A = 15^\circ\) by writing the multiplicative identity element 1 as \(\frac{1}{2}(2)\).

\[
\begin{align*}
\sin 15^\circ \cos 15^\circ &= \frac{1}{2}(2) \sin 15^\circ \cos 15^\circ & \text{Multiply by 1 in the form } \frac{1}{2}(2). \\
&= \frac{1}{2}(2 \sin 15^\circ \cos 15^\circ) & \text{Associative property (Section R.1)} \\
&= \frac{1}{2} \sin(2 \cdot 15^\circ) & 2 \sin A \cos A = \sin 2A, \text{ with } A = 15^\circ \\
&= \frac{1}{2} \sin 30^\circ \\
&= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} & \sin 30^\circ = \frac{1}{2} \text{ (Section 5.3)}
\end{align*}
\]

Now try Exercises 13 and 15.

Identities involving larger multiples of the variable can be derived by repeated use of the double-angle identities and other identities.

EXAMPLE 4  Deriving a Multiple-Angle Identity

Write \(\sin 3x\) in terms of \(\sin x\).

Solution

\[
\begin{align*}
\sin 3x &= \sin(2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x & \text{Sine sum identity (Section 7.3)} \\
&= (2 \sin x \cos x) \cos x + (\cos^2 x - \sin^2 x) \sin x & \text{Double-angle identities} \\
&= 2 \sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x & \text{Multiply.} \\
&= 2 \sin x(1 - \sin^2 x) + (1 - \sin^2 x) \sin x - \sin^3 x & \cos^2 x = 1 - \sin^2 x \\
&= 2 \sin x - 2 \sin^3 x + \sin x - \sin^3 x - \sin^3 x & \text{Distributive property (Section R.1)} \\
&= 3 \sin x - 4 \sin^3 x & \text{Combine terms.}
\end{align*}
\]

Now try Exercise 21.
The next example applies a multiple-angle identity to answer a question about electric current.

**EXAMPLE 5  Determining Wattage Consumption**

If a toaster is plugged into a common household outlet, the wattage consumed is not constant. Instead, it varies at a high frequency according to the model

\[ W = \frac{V^2}{R}, \]

where \( V \) is the voltage and \( R \) is a constant that measures the resistance of the toaster in ohms. (Source: Bell, D., *Fundamentals of Electric Circuits*, Fourth Edition, Prentice-Hall, 1998.) Graph the wattage \( W \) consumed by a typical toaster with \( R = 15 \) and \( V = 163 \sin 120\pi t \) in the window \([0, .05]\) by \([-500, 2000]\). How many oscillations are there?

**Solution**  Substituting the given values into the wattage equation gives

\[ W = \frac{V^2}{R} = \frac{(163 \sin 120\pi t)^2}{15}. \]

To determine the range of \( W \), we note that \( \sin 120\pi t \) has maximum value 1, so the expression for \( W \) has maximum value \( \frac{163^2}{15} = 1771 \). The minimum value is 0. The graph in Figure 6 shows that there are six oscillations.

Now try Exercise 81.

**Product-to-Sum and Sum-to-Product Identities**  Because they make it possible to rewrite a product as a sum, the identities for \( \cos(A + B) \) and \( \cos(A - B) \) are used to derive a group of identities useful in calculus.

Adding the identities for \( \cos(A + B) \) and \( \cos(A - B) \) gives

\[
\begin{align*}
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
\cos(A - B) &= \cos A \cos B + \sin A \sin B \\
\cos(A + B) + \cos(A - B) &= 2 \cos A \cos B
\end{align*}
\]

or

\[ \cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]. \]

Similarly, subtracting \( \cos(A + B) \) from \( \cos(A - B) \) gives

\[ \sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]. \]

Using the identities for \( \sin(A + B) \) and \( \sin(A - B) \) in the same way, we get two more identities. Those and the previous ones are now summarized.

**Product-to-Sum Identities**

\[
\begin{align*}
\cos A \cos B &= \frac{1}{2} [\cos(A + B) + \cos(A - B)] \\
\sin A \sin B &= \frac{1}{2} [\cos(A - B) - \cos(A + B)]
\end{align*}
\]
\[ \sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)] \]
\[ \cos A \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)] \]

**EXAMPLE 6** Using a Product-to-Sum Identity

Write \( \cos 2\theta \sin \theta \) as the sum or difference of two functions.

**Solution** Use the identity for \( \cos A \sin B \), with \( A = 2\theta \) and \( \theta = B \).

\[
\cos 2\theta \sin \theta = \frac{1}{2} [\sin(2\theta + \theta) - \sin(2\theta - \theta)]
\]
\[
= \frac{1}{2} \sin 3\theta - \frac{1}{2} \sin \theta
\]

Now try Exercise 41.

From these new identities we can derive another group of identities that are used to write sums of trigonometric functions as products.

**Sum-to-Product Identities**

\[
\sin A + \sin B = 2 \sin \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right)
\]
\[
\sin A - \sin B = 2 \cos \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right)
\]
\[
\cos A + \cos B = 2 \cos \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right)
\]
\[
\cos A - \cos B = -2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right)
\]

**EXAMPLE 7** Using a Sum-to-Product Identity

Write \( \sin 2\theta - \sin 4\theta \) as a product of two functions.

**Solution** Use the identity for \( \sin A - \sin B \), with \( 2\theta = A \) and \( 4\theta = B \).

\[
\sin 2\theta - \sin 4\theta = 2 \cos \left( \frac{2\theta + 4\theta}{2} \right) \sin \left( \frac{2\theta - 4\theta}{2} \right)
\]
\[
= 2 \cos \frac{6\theta}{2} \sin \left( \frac{-2\theta}{2} \right)
\]
\[
= 2 \cos 3\theta \sin(-\theta)
\]
\[
= -2 \cos 3\theta \sin \theta \quad \sin(-\theta) = -\sin \theta \text{ (Section 7.1)}
\]

Now try Exercise 45.
Half-Angle Identities  From the alternative forms of the identity for \( \cos 2A \), we derive three additional identities for \( \sin \frac{A}{2} \), \( \cos \frac{A}{2} \), and \( \tan \frac{A}{2} \). These are known as half-angle identities.

To derive the identity for \( \sin \frac{A}{2} \), start with the following double-angle identity for cosine and solve for \( \sin x \).

\[
\cos 2x = 1 - 2 \sin^2 x
\]
\[
2 \sin^2 x = 1 - \cos 2x
\]
\[
\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}
\]
Add 2 \( \sin^2 x \); subtract \( \cos 2x \). Divide by 2; take square roots. (Section 1.4)

Let \( 2x = A \), so \( x = \frac{A}{2} \); substitute.

The \( \pm \) sign in this identity indicates that the appropriate sign is chosen depending on the quadrant of \( \frac{A}{2} \). For example, if \( \frac{A}{2} \) is a quadrant III angle, we choose the negative sign since the sine function is negative in quadrant III.

We derive the identity for \( \cos \frac{A}{2} \) using the double-angle identity \( \cos 2x = 2 \cos^2 x - 1 \).

\[
1 + \cos 2x = 2 \cos^2 x
\]
Add 1.
\[
\cos^2 x = \frac{1 + \cos 2x}{2}
\]
Rewrite; divide by 2.
\[
\cos x = \pm \sqrt{\frac{1 + \cos 2x}{2}}
\]
Take square roots.
\[
\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}
\]
Replace \( x \) with \( \frac{A}{2} \).

An identity for \( \tan \frac{A}{2} \) comes from the identities for \( \sin \frac{A}{2} \) and \( \cos \frac{A}{2} \).

\[
\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\pm \sqrt{\frac{1 - \cos A}{2}}}{\sqrt{\frac{1 + \cos A}{2}}} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}}
\]

We derive an alternative identity for \( \tan \frac{A}{2} \) using double-angle identities.

\[
\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2}}
\]
Multiply by 2 \( \cos \frac{A}{2} \) in numerator and denominator.
\[
= \frac{\sin 2\left(\frac{A}{2}\right)}{1 + \cos 2\left(\frac{A}{2}\right)}
\]
Double-angle identities
\[
\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}
\]

From this identity for \( \tan \frac{A}{2} \), we can also derive

\[
\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}.
\]
7.4 Double-Angle Identities and Half-Angle Identities

Half-Angle Identities

\[
\begin{align*}
\cos \left( \frac{A}{2} \right) &= \pm \sqrt{\frac{1 + \cos A}{2}} \\
\sin \left( \frac{A}{2} \right) &= \pm \sqrt{\frac{1 - \cos A}{2}} \\
\tan \left( \frac{A}{2} \right) &= \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \\
\tan \left( \frac{A}{2} \right) &= \frac{\sin A}{1 + \cos A} \\
\tan \left( \frac{A}{2} \right) &= \frac{1 - \cos A}{\sin A}
\end{align*}
\]

**TEACHING TIP** Point out that the first identity for \( \tan \left( \frac{A}{2} \right) \) follows directly from the cosine and sine half-angle identities; however, the other two identities for \( \tan \left( \frac{A}{2} \right) \) are more useful.

**NOTE** The last two identities for \( \tan \left( \frac{A}{2} \right) \) do not require a sign choice. When using the other half-angle identities, select the plus or minus sign according to the quadrant in which \( A \) terminates. For example, if an angle \( A \) then \( \frac{A}{2} \), which lies in quadrant II. In quadrant II, \( \cos \frac{A}{2} \) and \( \tan \frac{A}{2} \) are negative, while \( \sin \frac{A}{2} \) is positive.

**EXAMPLE 8** Using a Half-Angle Identity to Find an Exact Value

Find the exact value of \( \cos 15^\circ \) using the half-angle identity for cosine.

**Solution**

\[
\cos 15^\circ = \cos \left( \frac{1}{2} (30^\circ) \right) = \sqrt{ \frac{1 + \cos 30^\circ}{2} }
\]

Choose the positive square root.

\[
= \sqrt{ \frac{1 + \frac{\sqrt{3}}{2}}{2} } = \sqrt{ \frac{2 + \sqrt{3}}{2} } \
\]

Simplify the radicals. (Section R.7)

Now try Exercise 51.

**EXAMPLE 9** Using a Half-Angle Identity to Find an Exact Value

Find the exact value of \( \tan 22.5^\circ \) using the identity \( \tan \left( \frac{A}{2} \right) = \frac{\sin A}{1 + \cos A} \).

**Solution** Since \( 22.5^\circ = \frac{1}{2} (45^\circ) \), replace \( A \) with \( 45^\circ \).

\[
\tan 22.5^\circ = \tan \left( \frac{45^\circ}{2} \right) = \frac{\sin 45^\circ}{1 + \cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}
\]

Now multiply numerator and denominator by 2. Then rationalize the denominator.

\[
\tan 22.5^\circ = \frac{\sqrt{2}}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = \frac{\sqrt{2} - 2}{2}
\]

\[
= \frac{2(\sqrt{2} - 1)}{2} = \sqrt{2} - 1
\]

Now try Exercise 53.
EXAMPLE 10  Finding Functions of $\frac{s}{2}$ Given Information about $s$

Given $\cos s = \frac{2}{3}$, with $\frac{3\pi}{2} < s < 2\pi$, find $\cos \frac{s}{2}$, $\sin \frac{s}{2}$, and $\tan \frac{s}{2}$.

**Solution**  Since

$$\frac{3\pi}{2} < s < 2\pi$$

and

$$\frac{3\pi}{4} < \frac{s}{2} < \pi, \quad \text{(Section 1.7)}$$

$s$ terminates in quadrant II. See Figure 7. In quadrant II, the values of $\cos \frac{s}{2}$ and $\tan \frac{s}{2}$ are negative and the value of $\sin \frac{s}{2}$ is positive. Now use the appropriate half-angle identities and simplify the radicals.

$$\sin \frac{s}{2} = \sqrt{1 - \frac{2}{3}} = \sqrt{\frac{1}{6}} = \frac{\sqrt{6}}{6}$$

$$\cos \frac{s}{2} = -\sqrt{1 + \frac{2}{3}} = -\sqrt{\frac{5}{6}} = -\frac{\sqrt{30}}{6}$$

$$\tan \frac{s}{2} = \frac{\sin \frac{s}{2}}{\cos \frac{s}{2}} = \frac{\sqrt{6}}{\sqrt{30}} = -\frac{\sqrt{5}}{5}$$

Notice that it is not necessary to use a half-angle identity for $\tan \frac{s}{2}$ once we find $\sin \frac{s}{2}$ and $\cos \frac{s}{2}$. However, using this identity would provide an excellent check.

Now try Exercise 59.

EXAMPLE 11  Simplifying Expressions Using the Half-Angle Identities

Simplify each expression.

(a) $\pm \sqrt{\frac{1 + \cos 12x}{2}}$

(b) $\frac{1 - \cos 5\alpha}{\sin 5\alpha}$

**Solution**

(a) This matches part of the identity for $\cos \frac{A}{2}$.

$$\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}$$

Replace $A$ with $12x$ to get

$$\pm \sqrt{\frac{1 + \cos 12x}{2}} = \cos \frac{12x}{2} = \cos 6x.$$  

(b) Use the third identity for $\tan \frac{A}{2}$ given earlier with $A = 5\alpha$ to get

$$\frac{1 - \cos 5\alpha}{\sin 5\alpha} = \frac{5\alpha}{2}.$$  

Now try Exercises 67 and 71.
7.4 Exercises

Use identities to find values of the sine and cosine functions for each angle measure. See Example 1.

1. \( \cos \theta = \frac{2\sqrt{5}}{5} \); \( \sin \theta = \frac{\sqrt{5}}{5} \)
2. \( \cos \theta = -\frac{\sqrt{14}}{4} \); \( \sin \theta = -\frac{\sqrt{2}}{4} \)
3. \( \cos x = -\frac{\sqrt{42}}{12} \); \( \sin x = \frac{\sqrt{102}}{12} \)
4. \( \cos x = -\frac{\sqrt{30}}{6} \); \( \sin x = \frac{\sqrt{6}}{6} \)
5. \( \cos 2\theta = \frac{19}{25} \)
6. \( \sin 2\theta = -\frac{4\sqrt{21}}{25} \)
7. \( \cos 2\theta = \frac{119}{169} \); \( \sin 2\theta = -\frac{120}{169} \)
8. \( \cos 2x = -\frac{3}{5} \); \( \sin 2x = \frac{4}{5} \)
9. \( \cos 2\theta = \frac{39}{49} \)
10. \( \sin 2\theta = -\frac{4\sqrt{55}}{49} \)

Use an identity to write each expression as a single trigonometric function value or as a single number. See Example 3.

11. \( \cos 15^\circ - \sin 15^\circ \)
12. \( \frac{2 \tan 15^\circ}{1 - \tan^2 15^\circ} \)
13. \( 1 - 2 \sin^2 15^\circ \)
14. \( 1 - 2 \sin^2 22\frac{1}{2}^\circ \)
15. \( 2 \cos^2 67\frac{1}{2}^\circ - 1 \)
16. \( \cos^2 \frac{\pi}{8} - \frac{1}{2} \)
17. \( \frac{\tan 51^\circ}{1 - \tan^2 51^\circ} \)
18. \( \frac{\tan 34^\circ}{2(1 - \tan^2 34^\circ)} \)
19. \( \frac{1}{4} - \frac{1}{2} \sin^2 47.1^\circ \)

Express each function as a trigonometric function of \( x \). See Example 4.

21. \( \cos 3x \)
22. \( \sin 4x \)
23. \( \tan 3x \)
24. \( \cos 4x \)

Graph each expression and use the graph to conjecture an identity. Then verify your conjecture algebraically.

25. \( \cos^4 x - \sin^4 x \)
26. \( \frac{4 \tan x \cos^2 x - 2 \tan x}{1 - \tan^2 x} \)

Verify that each equation is an identity. See Example 2.

27. \( (\sin x + \cos x)^2 = \sin 2x + 1 \)
28. \( \sec 2x = \frac{\sec^2 x + \sec^4 x}{2 + \sec^2 x - \sec^4 x} \)
### Trigonometric Identities and Equations

<table>
<thead>
<tr>
<th>Problem</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>41.</td>
<td>$\sin 160^\circ - \sin 44^\circ$</td>
</tr>
<tr>
<td>42.</td>
<td>$\sin 225^\circ + \sin 55^\circ$</td>
</tr>
<tr>
<td>43.</td>
<td>$\frac{5}{2} \cos 5x + \frac{5}{2} \cos x$</td>
</tr>
<tr>
<td>44.</td>
<td>$\frac{1}{2} \cos x - \frac{1}{2} \cos 9x$</td>
</tr>
<tr>
<td>45.</td>
<td>$\sin 3x \sin x$</td>
</tr>
<tr>
<td>46.</td>
<td>$2 \cos 6.5^\circ \cos 1.5^\circ$</td>
</tr>
<tr>
<td>47.</td>
<td>$\sin 11.5^\circ \cos 36.5^\circ$</td>
</tr>
<tr>
<td>48.</td>
<td>$2 \cos 98.5^\circ \sin 3.5^\circ$</td>
</tr>
<tr>
<td>49.</td>
<td>$2 \cos 6 \sin 2x$</td>
</tr>
<tr>
<td>50.</td>
<td>$2 \cos 6 \sin 3x$</td>
</tr>
<tr>
<td>51.</td>
<td>$\sqrt{2 + \sqrt{2}}$</td>
</tr>
<tr>
<td>52.</td>
<td>$\frac{\sqrt{2} - \sqrt{3}}{2}$</td>
</tr>
<tr>
<td>53.</td>
<td>$\frac{\sqrt{2} + \sqrt{3}}{2}$</td>
</tr>
<tr>
<td>54.</td>
<td>$2 - \sqrt{3}$</td>
</tr>
<tr>
<td>55.</td>
<td>$\frac{\sqrt{2} + \sqrt{3}}{2}$</td>
</tr>
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<td>56.</td>
<td>$\frac{\sqrt{2} - \sqrt{3}}{2}$</td>
</tr>
<tr>
<td>57.</td>
<td>$\frac{\sqrt{10}}{4}$</td>
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<tr>
<td>58.</td>
<td>$\frac{\sqrt{13}}{4}$</td>
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<tr>
<td>59.</td>
<td>$\frac{\sqrt{50} - 20\sqrt{6}}{10}$</td>
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<tr>
<td>60.</td>
<td>$\frac{\sqrt{50} - 10\sqrt{5}}{10}$</td>
</tr>
<tr>
<td>61.</td>
<td>$\frac{\sqrt{50} - 15\sqrt{10}}{10}$</td>
</tr>
<tr>
<td>62.</td>
<td>$\frac{\sqrt{7}}{6}$</td>
</tr>
<tr>
<td>63.</td>
<td>$\frac{\sqrt{3}}{5}$</td>
</tr>
</tbody>
</table>

#### Write each expression as a sum or difference of trigonometric functions. See Example 6.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>41.</td>
<td>$2 \sin 58^\circ \cos 102^\circ$</td>
</tr>
<tr>
<td>42.</td>
<td>$2 \cos 85^\circ \sin 140^\circ$</td>
</tr>
<tr>
<td>43.</td>
<td>$5 \cos 3x \cos 2x$</td>
</tr>
<tr>
<td>44.</td>
<td>$4 \sin x \sin 5x$</td>
</tr>
</tbody>
</table>

#### Write each expression as a product of trigonometric functions. See Example 7.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>45.</td>
<td>$4 \cos x - \cos 2x$</td>
</tr>
<tr>
<td>46.</td>
<td>$5 \cos x + \cos 8x$</td>
</tr>
<tr>
<td>47.</td>
<td>$25^\circ + \sin(-48^\circ)$</td>
</tr>
<tr>
<td>48.</td>
<td>$\sin 102^\circ - \sin 95^\circ$</td>
</tr>
<tr>
<td>49.</td>
<td>$\cos 4x + \cos 8x$</td>
</tr>
<tr>
<td>50.</td>
<td>$\sin 9x - \sin 3x$</td>
</tr>
</tbody>
</table>

#### Use a half-angle identity to find each exact value. See Examples 8 and 9.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>51.</td>
<td>$\sin 67.5^\circ$</td>
</tr>
<tr>
<td>52.</td>
<td>$\sin 195^\circ$</td>
</tr>
<tr>
<td>53.</td>
<td>$\cos 195^\circ$</td>
</tr>
<tr>
<td>54.</td>
<td>$\tan 195^\circ$</td>
</tr>
<tr>
<td>55.</td>
<td>$\cos 165^\circ$</td>
</tr>
<tr>
<td>56.</td>
<td>$\sin 165^\circ$</td>
</tr>
</tbody>
</table>

#### Find each of the following. See Example 10.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>59.</td>
<td>$\cos \frac{x}{2}$, given $\cos x = \frac{1}{4}$, with $0 &lt; x &lt; \frac{\pi}{2}$</td>
</tr>
<tr>
<td>60.</td>
<td>$\sin \frac{x}{2}$, given $\cos x = -\frac{5}{8}$, with $\frac{\pi}{2} &lt; x &lt; \pi$</td>
</tr>
<tr>
<td>61.</td>
<td>$\tan \frac{\theta}{2}$, given $\sin \theta = \frac{3}{5}$, with $90^\circ &lt; \theta &lt; 180^\circ$</td>
</tr>
<tr>
<td>62.</td>
<td>$\cos \frac{\theta}{2}$, given $\sin \theta = -\frac{1}{5}$, with $180^\circ &lt; \theta &lt; 270^\circ$</td>
</tr>
<tr>
<td>63.</td>
<td>$\sin \frac{x}{2}$, given $\tan x = 2$, with $0 &lt; x &lt; \frac{\pi}{2}$</td>
</tr>
<tr>
<td>64.</td>
<td>$\cos \frac{x}{2}$, given $\cot x = -3$, with $\frac{\pi}{2} &lt; x &lt; \pi$</td>
</tr>
<tr>
<td>65.</td>
<td>$\tan \frac{\theta}{2}$, given $\tan \theta = \frac{\sqrt{7}}{3}$, with $180^\circ &lt; \theta &lt; 270^\circ$</td>
</tr>
<tr>
<td>66.</td>
<td>$\cot \frac{\theta}{2}$, given $\tan \theta = -\frac{\sqrt{3}}{2}$, with $90^\circ &lt; \theta &lt; 180^\circ$</td>
</tr>
</tbody>
</table>
7.4 Double-Angle Identities and Half-Angle Identities 643

Use an identity to write each expression with a single trigonometric function. See Example 11.

67. \( \sqrt{\frac{1 - \cos 40^\circ}{2}} \) 68. \( \sqrt{\frac{1 + \cos 76^\circ}{2}} \) 69. \( \sqrt{\frac{1 - \cos 147^\circ}{1 + \cos 147^\circ}} \)

70. \( \sqrt{\frac{1 + \cos 165^\circ}{1 - \cos 165^\circ}} \) 71. \( \frac{1 - \cos 59.74^\circ}{\sin 59.74^\circ} \) 72. \( \frac{\sin 158.2^\circ}{1 + \cos 158.2^\circ} \)

73. Use the identity \( \tan \frac{A}{4} = \frac{\sin A}{1 + \cos A} \) to derive the equivalent identity \( \tan \frac{A}{2} = \frac{1 - \cos A}{\sin A} \) by multiplying both the numerator and denominator by \( 1 - \cos A \).

74. Consider the expression \( \tan \left( \frac{x}{2} + x \right) \).

(a) Why can’t we use the identity for \( \tan (A + B) \) to express it as a function of \( x \) alone?
(b) Use the identity \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) to rewrite the expression in terms of sine and cosine.
(c) Use the result of part (b) to show that \( \tan \left( \frac{x}{2} + x \right) = -\cot x \).

**Modeling)** Mach Number An airplane flying faster than sound sends out sound waves that form a cone, as shown in the figure. The cone intersects the ground to form a hyperbola. As this hyperbola passes over a particular point on the ground, a sonic boom is heard at that point. If \( \theta \) is the angle at the vertex of the cone, then

\[
\frac{\sin \frac{\theta}{2}}{2} = \frac{1}{m},
\]

where \( m \) is the Mach number for the speed of the plane. (We assume \( m > 1 \).) The Mach number is the ratio of the speed of the plane and the speed of sound. Thus, a speed of Mach 1.4 means that the plane is flying at 1.4 times the speed of sound. In Exercises 75–78, one of the values \( \theta \) or \( m \) is given. Find the other value.

75. \( m = \frac{3}{2} \) 76. \( m = \frac{5}{4} \) 77. \( \theta = 30^\circ \) 78. \( \theta = 60^\circ \)

**Modeling)** Solve each problem. See Example 5.

79. Railroad Curves In the United States, circular railroad curves are designated by the degree of curvature, the central angle subtended by a chord of 100 ft. See the figure. (Source: Hay, W. W., *Railroad Engineering*, John Wiley & Sons, 1982.)

(a) Use the figure to write an expression for \( \cos \frac{\theta}{2} \).
(b) Use the result of part (a) and the third half-angle identity for tangent to write an expression for \( \tan \frac{\theta}{2} \).
(c) If \( b = 12 \), what is the measure of angle \( \theta \) to the nearest degree?

80. Distance Traveled by a Stone The distance \( D \) of an object thrown (or propelled) from height \( h \) (feet) at angle \( \theta \) with initial velocity \( v \) is modeled by the formula

\[
D = \frac{v^2 \sin \theta \cos \theta + v \cos \theta \sqrt{(v \sin \theta)^2 + 64h}}{32}.
\]

See the figure. (Source: Kreighbaum, E., and K. Barthels, *Biomechanics*, Allyn & Bacon, 1996.) Also see the Chapter 5 Quantitative Reasoning.

(a) Find \( D \) when \( h = 0 \); that is, when the object is propelled from the ground.
(b) Suppose a car driving over loose gravel kicks up a small stone at a velocity of 36 ft per sec (about 25 mph) and an angle \( \theta = 30^\circ \). How far will the stone travel?
Summary Exercises on Verifying Trigonometric Identities

These summary exercises provide practice with the various types of trigonometric identities presented in this chapter. Verify that each equation is an identity.

1. \( \tan \theta + \cot \theta = \sec \theta \csc \theta \)
2. \( \csc \theta \cos^2 \theta + \sin \theta = \csc \theta \)
3. \( \tan \frac{x}{2} = \sec x - \cot x \)
4. \( \sec(\pi - x) = -\sec x \)
5. \( \sin t = \frac{1 - \cos t}{1 + \cos t} \)
6. \( \frac{1 - \sin t}{\cos t} = \frac{1}{\sec t + \tan t} \)
7. \( \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \)
8. \( \frac{2}{1 + \cos x} - \tan \frac{x}{2} = 1 \)
9. \( \cot \theta - \tan \theta = \frac{2 \cos^2 \theta - 1}{\sin \theta \cos \theta} \)
10. \( \frac{1}{\sec t - 1} + \frac{1}{\sec t + 1} = 2 \cot t \csc t \)
11. \( \sin(x + y) = \cot x \cot y = \frac{\cos(x + y)}{1 + \cot x \cot y} \)
12. \( 1 - \tan^2 \theta = \frac{2 \cos \theta}{1 + \cos \theta} \)
13. \( \sin \theta + \tan \frac{\theta}{1 + \cos \theta} = \tan \theta \)
14. \( \csc^4 x - \cot^4 x = \frac{1 + \cos^2 x}{1 - \cos^2 x} \)
15. \( \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \)
16. \( \cos 2x = \frac{2 - \sec^2 x}{\sec^2 x} \)
17. \( \tan^2 t + \frac{1}{\tan t \csc^2 t} = \tan t \)
18. \( \sin s \frac{1 + \cos s}{1 + \cos s} + \frac{1 + \cos s}{\sin s} = 2 \csc s \)
19. \( \tan \frac{x}{2} + \frac{\pi}{4} = \sec x + \tan x \)
20. \( \tan \frac{x}{2} + \cot \theta = \frac{1 - 2 \cos^2 \theta}{\tan \theta + \cot \theta} \)
23. \( \tan(x + y) - \tan y = \tan x \)
\[
\frac{\cos^2 x - \sin^2 x}{\cos^2 x} = 1 - \tan^2 x
\]
25. \( \frac{2(\sin x - \sin^3 x)}{\cos x} = \sin 2x \)
24. \( 2 \cos^2 \frac{x}{2} \tan x = \tan x + \sin x \)
26. \( \frac{\csc t + 1}{\csc t - 1} = (\sec t + \tan t)^2 \)
28. \( \frac{1}{2} \cot \frac{x}{2} - \frac{1}{2} \tan \frac{x}{2} = \cot x \)

### 7.5 Inverse Circular Functions

**Inverse Functions**

We first discussed inverse functions in Section 4.1. We give a quick review here for a pair of inverse functions \( f \) and \( f^{-1} \).

1. If a function \( f \) is one-to-one, then \( f \) has an inverse function \( f^{-1} \).
2. In a one-to-one function, each \( x \)-value corresponds to only one \( y \)-value and each \( y \)-value corresponds to only one \( x \)-value.
3. The domain of \( f \) is the range of \( f^{-1} \), and the range of \( f \) is the domain of \( f^{-1} \).
4. The graphs of \( f \) and \( f^{-1} \) are reflections of each other about the line \( y = x \).
5. To find \( f^{-1}(x) \) from \( f(x) \), follow these steps.
   - **Step 1** Replace \( f(x) \) with \( y \) and interchange \( x \) and \( y \).
   - **Step 2** Solve for \( y \).
   - **Step 3** Replace \( y \) with \( f^{-1}(x) \).

In the remainder of this section, we use these facts to develop the inverse circular (trigonometric) functions.

**Inverse Sine Function**

From Figure 8 and the horizontal line test, we see that \( y = \sin x \) does not define a one-to-one function. If we restrict the domain to the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), which is the part of the graph in Figure 8 shown in color, this restricted function is one-to-one and has an inverse function. The range of \( y = \sin x \) is \( [-1, 1] \), so the domain of the inverse function will be \( [-1, 1] \), and its range will be \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).

Looking Ahead to Calculus

The inverse circular functions are used in calculus to solve certain types of related-rates problems and to integrate certain rational functions.

**Teaching Tip**

Mention that the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) contains enough of the graph of the sine function to include all possible values of \( y \). While other intervals could also be used, this interval is an accepted convention that is adopted by scientific calculators and graphing calculators.

![Figure 8](image-url)
Reflecting the graph of \( y = \sin x \) on the restricted domain across the line \( y = x \) gives the graph of the inverse function, shown in Figure 9. Some key points are labeled on the graph. The equation of the inverse of \( y = \sin x \) is found by interchanging \( x \) and \( y \) to get \( x = \sin y \). This equation is solved for \( y \) by writing \( y = \sin^{-1} x \) (read “inverse sine of \( x \)”). As Figure 9 shows, the domain of \( y = \sin^{-1} x \) is \([-1, 1]\), while the restricted domain of \( y = \sin x \), \([-\frac{\pi}{2}, \frac{\pi}{2}]\), is the range of \( y = \sin^{-1} x \). An alternative notation for \( \sin^{-1} x \) is \( \arcsin x \).

### Inverse Sine Function

\[ y = \sin^{-1} x \text{ or } y = \arcsin x \]

means that \( x = \sin y \), for \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\).

We can think of \( y = \sin^{-1} x \) or \( y = \arcsin x \) as “\( y \) is the number in the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\) whose sine is \( x \).” Just as we evaluated \( y = \log_2 4 \) by writing it in exponential form as \( 2^y = 4 \) (Section 4.3), we can write \( y = \sin^{-1} x \) as \( \sin y = x \) to evaluate it. We must pay close attention to the domain and range intervals.

#### EXAMPLE 1 Finding Inverse Sine Values

Find \( y \) in each equation.

**(a)** \( y = \arcsin \frac{1}{2} \)

**Algebraic Solution**

(a) The graph of the function defined by \( y = \arcsin x \) (Figure 9) includes the point \((\frac{1}{2}, \frac{\pi}{6})\). Thus,

\[ \arcsin \frac{1}{2} = \frac{\pi}{6}. \]

Alternatively, we can think of \( y = \arcsin \frac{1}{2} \) as “\( y \) is the number in \([-\frac{\pi}{2}, \frac{\pi}{2}]\) whose sine is \( \frac{1}{2} \).” Then we can write the given equation as \( \sin y = \frac{1}{2} \). Since \( \sin \frac{\pi}{6} = \frac{1}{2} \) and \( \frac{\pi}{6} \) is in the range of the arcsine function, \( y = \frac{\pi}{6} \).

**(b)** Writing the equation \( y = \sin^{-1}(-1) \) in the form \( \sin y = -1 \) shows that \( y = -\frac{\pi}{2} \). This can be verified by noticing that the point \((-1, -\frac{\pi}{2})\) is on the graph of \( y = \sin^{-1} x \).

**(c)** Because \(-2\) is not in the domain of the inverse sine function, \( \sin^{-1}(-2) \) does not exist.

**Graphing Calculator Solution**

To find these values with a graphing calculator, we graph \( Y_1 = \sin^{-1} X \) and locate the points with X-values \( \frac{\pi}{6} \) and \(-1\). Figure 10(a) shows that when \( X = \frac{\pi}{6} \), \( Y = \frac{\pi}{6} = .52359878 \). Similarly, Figure 10(b) shows that when \( X = -1 \), \( Y = -\frac{\pi}{2} \approx -1.570796 \).

Since \( \sin^{-1}(-2) \) does not exist, a calculator will give an error message for this input.

**Now try Exercises 13 and 23.**

**CAUTION** In Example 1(b), it is tempting to give the value of \( \sin^{-1}(-1) \) as \( \frac{3\pi}{2} \), since \( \sin \frac{3\pi}{2} = -1 \). Notice, however, that \( \frac{3\pi}{2} \) is not in the range of the inverse sine function. Be certain that the number given for an inverse function value is in the range of the particular inverse function being considered.
Our observations about the inverse sine function from Figure 9 lead to the following generalizations.

**Inverse Sine Function**

\[ y = \sin^{-1}x \quad \text{or} \quad y = \arcsin x \]

**Domain:** \([-1, 1] \]

**Range:** \([-\frac{\pi}{2}, \frac{\pi}{2}] \]

- The inverse sine function is increasing and continuous on its domain \([-1, 1]\).
- Its \(x\)-intercept is 0, and its \(y\)-intercept is 0.
- Its graph is symmetric with respect to the origin; it is an odd function.

**Inverse Cosine Function**

The function \(y = \cos^{-1}x\) (or \(y = \arccos x\)) is defined by restricting the domain of the function \(y = \cos x\) to the interval \([0, \pi]\) as in Figure 12, and then interchanging the roles of \(x\) and \(y\). The graph of \(y = \cos^{-1}x\) is shown in Figure 13. Again, some key points are shown on the graph.

**Inverse Cosine Function**

\(y = \cos^{-1}x\) or \(y = \arccos x\) means that \(x = \cos y\), for \(0 \leq y \leq \pi\).
EXAMPLE 2  Finding Inverse Cosine Values

Find \( y \) in each equation.

(a) \( y = \arccos 1 \)

(b) \( y = \cos^{-1}\left( -\frac{\sqrt{2}}{2} \right) \)

Solution

(a) Since the point (1, 0) lies on the graph of \( y = \arccos x \) in Figure 13 on the previous page, the value of \( y \) is 0. Alternatively, we can think of \( y = \arccos 1 \) as “\( y \) is the number in \([0, \pi]\) whose cosine is 1,” or \( \cos y = 1 \). Then \( y = 0 \), since \( \cos 0 = 1 \) and 0 is in the range of the arccosine function.

(b) We must find the value of \( y \) that satisfies \( \cos y = -\frac{\sqrt{2}}{2} \), where \( y \) is in the interval \([0, \pi]\), the range of the function \( y = \cos^{-1} x \). The only value for \( y \) that satisfies these conditions is \( \frac{3\pi}{4} \). Again, this can be verified from the graph in Figure 13.

Now try Exercises 15 and 21.

Our observations about the inverse cosine function lead to the following generalizations.

**INVERSE COSINE FUNCTION**

\[ y = \cos^{-1} x \quad \text{or} \quad y = \arccos x \]

- Domain: \([-1, 1]\)  
- Range: \([0, \pi]\)

![Graph of \( y = \cos^{-1} x \)]

- The inverse cosine function is decreasing and continuous on its domain \([-1, 1]\).
- Its \( x \)-intercept is 1, and its \( y \)-intercept is \( \frac{\pi}{2} \).
- Its graph is not symmetric with respect to the \( y \)-axis or the origin.

**Inverse Tangent Function**  Restricting the domain of the function \( y = \tan x \) to the open interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) yields a one-to-one function. By interchanging the roles of \( x \) and \( y \), we obtain the inverse tangent function given by \( y = \tan^{-1} x \) or \( y = \arctan x \). Figure 15 shows the graph of the restricted tangent function. Figure 16 gives the graph of \( y = \tan^{-1} x \).
### Inverse Tangent Function

\[
y = \tan^{-1}x \quad \text{or} \quad y = \arctan x
\]

means that \( x = \tan y \), for \(-\frac{\pi}{2} < y < \frac{\pi}{2}\).

---

**INVERSE TANGENT FUNCTION**

\[
y = \tan^{-1}x \quad \text{or} \quad y = \arctan x
\]

**Domain:** \((-\infty, \infty)\)  
**Range:** \((-\frac{\pi}{2}, \frac{\pi}{2})\)

- The inverse tangent function is increasing and continuous on its domain \((-\infty, \infty)\).
- Its \(x\)-intercept is 0, and its \(y\)-intercept is 0.
- Its graph is symmetric with respect to the origin; it is an odd function.
- The lines \(y = \frac{\pi}{2}\) and \(y = -\frac{\pi}{2}\) are horizontal asymptotes.

**Remaining Inverse Circular Functions** The remaining three inverse trigonometric functions are defined similarly; their graphs are shown in Figure 18. All six inverse trigonometric functions with their domains and ranges are given in the table on the next page.
650  CHAPTER 7  Trigonometric Identities and Equations

<table>
<thead>
<tr>
<th>Inverse Function</th>
<th>Domain</th>
<th>Interval</th>
<th>Quadrants of the Unit Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sin^{-1} x )</td>
<td>([-1, 1])</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
<td>I and IV</td>
</tr>
<tr>
<td>( y = \cos^{-1} x )</td>
<td>([-1, 1])</td>
<td>([0, \pi])</td>
<td>I and II</td>
</tr>
<tr>
<td>( y = \tan^{-1} x )</td>
<td>((-\infty, \infty))</td>
<td>((-\frac{\pi}{2}, \frac{\pi}{2}))</td>
<td>I and IV</td>
</tr>
<tr>
<td>( y = \cot^{-1} x )</td>
<td>((-\infty, \infty))</td>
<td>((0, \pi))</td>
<td>I and II</td>
</tr>
<tr>
<td>( y = \sec^{-1} x )</td>
<td>((-\infty, -1) \cup [1, \infty))</td>
<td>([0, \pi], y \neq \frac{\pi}{2})</td>
<td>I and II</td>
</tr>
<tr>
<td>( y = \csc^{-1} x )</td>
<td>((-\infty, -1) \cup [1, \infty))</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}], y \neq 0)</td>
<td>I and IV</td>
</tr>
</tbody>
</table>

**Inverse Function Values**  The inverse circular functions are formally defined with real number ranges. However, there are times when it may be convenient to find degree-measured angles equivalent to these real number values. It is also often convenient to think in terms of the unit circle and choose the inverse function values based on the quadrants given in the preceding table.

**EXAMPLE 3  Finding Inverse Function Values (Degree-Measured Angles)**

Find the *degree measure* of \( \theta \) in the following.

(a) \( \theta = \arctan 1 \)  
(b) \( \theta = \sec^{-1} 2 \)

**Solution**

(a) Here \( \theta \) must be in \((-90^\circ, 90^\circ)\), but since \( 1 > 0 \), \( \theta \) must be in quadrant I. The alternative statement, \( \tan \theta = 1 \), leads to \( \theta = 45^\circ \).

(b) Write the equation as \( \sec \theta = 2 \). For \( \sec^{-1} x \), \( \theta \) is in quadrant I or II. Because 2 is positive, \( \theta \) is in quadrant I and \( \theta = 60^\circ \), since \( \sec 60^\circ = 2 \). Note that \( 60^\circ \) (the degree equivalent of \( \frac{\pi}{3} \)) is in the range of the inverse secant function.

Now try Exercises 33 and 39.

The inverse trigonometric function keys on a calculator give results in the proper quadrant for the inverse sine, inverse cosine, and inverse tangent functions, according to the definitions of these functions. For example, on a calculator, in degrees, \( \sin^{-1} .5 = 30^\circ \), \( \sin^{-1}(-.5) = -30^\circ \), \( \tan^{-1}(-1) = -45^\circ \), and \( \cos^{-1}(-.5) = 120^\circ \).

Finding \( \cot^{-1} x \), \( \sec^{-1} x \), and \( \csc^{-1} x \) with a calculator is not as straightforward, because these functions must be expressed in terms of \( \tan^{-1} x \), \( \cos^{-1} x \), and \( \sin^{-1} x \), respectively. If \( y = \sec^{-1} x \), for example, then \( \sec y = x \), which must be written as a cosine function as follows:

If \( \sec y = x \), then \( \frac{1}{\cos y} = x \) or \( \cos y = \frac{1}{x} \), and \( y = \cos^{-1} \frac{1}{x} \).

*The inverse secant and inverse cosecant functions are sometimes defined with different ranges. We use intervals that match their reciprocal functions (except for one missing point).*
In summary, to find $\sec^{-1} x$, we find $\cos^{-1} \left(\frac{1}{x}\right)$. Similar statements apply to $\csc^{-1} x$ and $\cot^{-1} x$. There is one additional consideration with $\cot^{-1} x$. Since we take the inverse tangent of the reciprocal to find inverse cotangent, the calculator gives values of inverse cotangent with the same range as inverse tangent, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is not the correct range for inverse cotangent. For inverse cotangent, the proper range must be considered and the results adjusted accordingly.

**EXAMPLE 4** Finding Inverse Function Values with a Calculator

(a) Find $y$ in radians if $y = \csc^{-1}(-3)$.
(b) Find $\theta$ in degrees if $\theta = \arccot(-.3541)$.

**Solution**

(a) With the calculator in radian mode, enter $\csc^{-1}(-3)$ as $\sin^{-1}\left(\frac{1}{-3}\right)$ to get $y = -\arcsin(\frac{1}{3})$. See Figure 19.

(b) Set the calculator to degree mode. A calculator gives the inverse tangent value of a negative number as a quadrant IV angle. The restriction on the range of arccotangent implies that $\theta$ must be in quadrant II, so enter $\arctan\left(-\frac{3}{5}\right)$ as $\arccot(-.3541)$ as $\tan^{-1}\left(\frac{1}{-\arccot(-.3541)}\right) + 180^\circ$.

As shown in Figure 19, $\theta = 109.4990544^\circ$.

Now try Exercises 43 and 49.

**EXAMPLE 5** Finding Function Values Using Definitions of the Trigonometric Functions

Evaluate each expression without using a calculator.

(a) $\sin\left(\tan^{-1}\frac{3}{2}\right)$
(b) $\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right)$

**Solution**

(a) Let $\theta = \tan^{-1}\frac{3}{2}$, so $\tan \theta = \frac{3}{2}$. The inverse tangent function yields values only in quadrants I and IV, and since $\frac{3}{2}$ is positive, $\theta$ is in quadrant I. Sketch $\theta$ in quadrant I, and label a triangle, as shown in Figure 20. By the Pythagorean theorem, the hypotenuse is $\sqrt{13}$. The value of sine is the quotient of the side opposite and the hypotenuse, so

$$\sin\left(\tan^{-1}\frac{3}{2}\right) = \sin \theta = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}. \quad (\text{Section 5.3})$$

(b) Let $A = \cos^{-1}\left(-\frac{5}{13}\right)$. Then, $\cos A = -\frac{5}{13}$. Since $\cos^{-1} x$ for a negative value of $x$ is in quadrant II, sketch $A$ in quadrant II, as shown in Figure 21. 

$$\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right) = \tan A = -\frac{12}{5}$$

Now try Exercises 63 and 65.
TEACHING TIP  Point out that the equations \( \sin^{-1}(x) = \) \( x \),
\( \cos^{-1}(x) = x \), and
\( \tan^{-1}(x) = x \) are true wherever they are defined. However,\
\( \sin^{-1}(\sin x) = x \), \( \cos^{-1}(\cos x) = x \), and
\( \tan^{-1}(\tan x) = x \) are true only for values of \( x \) in the restricted
domains of the sine, cosine, and tangent functions.

EXAMPLE 6 Finding Function Values Using Identities

Evaluate each expression without using a calculator.

(a) \( \cos\left(\arctan\sqrt{3} + \arcsin\frac{1}{3}\right) \)   
(b) \( \tan\left(2\arcsin\frac{2}{5}\right) \)

Solution

(a) Let \( A = \arctan\sqrt{3} \) and \( B = \arcsin\frac{1}{3} \), so \( \tan A = \sqrt{3} \) and \( \sin B = \frac{1}{3} \). Sketch both \( A \) and \( B \) in quadrant I, as shown in Figure 22. Now, use the cosine sum identity.

\[
\cos(A + B) = \cos A \cos B - \sin A \sin B \quad \text{(Section 7.3)}
\]

\[
\cos\left(\arctan\sqrt{3} + \arcsin\frac{1}{3}\right) = \cos\left(\arctan\sqrt{3}\right) \cos\left(\arcsin\frac{1}{3}\right) - \sin\left(\arctan\sqrt{3}\right) \sin\left(\arcsin\frac{1}{3}\right) \quad (1)
\]

From Figure 22,

\[
\cos\left(\arctan\sqrt{3}\right) = \cos A = \frac{1}{2}, \quad \cos\left(\arcsin\frac{1}{3}\right) = \cos B = \frac{2\sqrt{2}}{3}.
\]

\[
\sin\left(\arctan\sqrt{3}\right) = \sin A = \frac{\sqrt{3}}{2}, \quad \sin\left(\arcsin\frac{1}{3}\right) = \sin B = \frac{1}{3}.
\]

Substitute these values into equation (1) to get

\[
\cos\left(\arctan\sqrt{3} + \arcsin\frac{1}{3}\right) = \frac{1}{2} \cdot \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2} \cdot \frac{1}{3} = \frac{2\sqrt{2} - \sqrt{3}}{6}.
\]

(b) Let \( \arcsin\frac{2}{5} = B \). Then, from the double-angle tangent identity,

\[
\tan\left(2\arcsin\frac{2}{5}\right) = \tan 2B = \frac{2 \tan B}{1 - \tan^2 B} \quad \text{(Section 7.4)}
\]

Since \( \arcsin\frac{2}{5} = B \), \( \sin B = \frac{2}{5} \). Sketch a triangle in quadrant I, find the
length of the third side, and then find \( \tan B \). From the triangle in Figure 23,\( \tan B = \frac{2}{\sqrt{21}} \), and

\[
\tan\left(2\arcsin\frac{2}{5}\right) = \frac{2\left(\frac{2}{\sqrt{21}}\right)}{1 - \left(\frac{2}{\sqrt{21}}\right)^2} = \frac{4}{21} \cdot \frac{1 - \frac{4}{21}}{17} = \frac{4\sqrt{21}}{17}.
\]

Now try Exercises 69 and 75.

While the work shown in Examples 5 and 6 does not rely on a calculator, we
can support our algebraic work with one. By entering \( \cos\left(\arctan\sqrt{3} + \arcsin\frac{1}{3}\right) \)
from Example 6(a) into a calculator, we get the approximation \( 0.1827293862 \), the
same approximation as when we enter \( \frac{2\sqrt{2} - \sqrt{3}}{6} \) (the exact value we obtained
algebraically). Similarly, we obtain the same approximation when we evaluate
\( \tan\left(2\arcsin\frac{2}{5}\right) \) and \( \frac{4\sqrt{21}}{17} \), supporting our answer in Example 6(b).
EXAMPLE 7  Writing Function Values in Terms of \( u \)

Write each trigonometric expression as an algebraic expression in \( u \).

(a) \( \sin(\tan^{-1} u) \)  
(b) \( \cos(2 \sin^{-1} u) \)

Solution (a) Let \( \theta = \tan^{-1} u \), so \( \tan \theta = u \). Here, \( u \) may be positive or negative. Since \( -\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2} \), sketch \( \theta \) in quadrants I and IV and label two triangles, as shown in Figure 24. Since sine is given by the quotient of the side opposite and the hypotenuse,

\[
\sin(\tan^{-1} u) = \sin \theta = \frac{u}{\sqrt{u^2 + 1}} = \frac{u\sqrt{u^2 + 1}}{u^2 + 1}.
\]

The result is positive when \( u \) is positive and negative when \( u \) is negative.

(b) Let \( \theta = \sin^{-1} u \), so \( \sin \theta = u \). To find \( \cos 2\theta \), use the identity \( \cos 2\theta = 1 - 2 \sin^2 \theta \).

\[
\cos(2 \sin^{-1} u) = \cos(2\theta) = 1 - 2 \sin^2 \theta = 1 - 2u^2
\]

Now try Exercises 83 and 85.

EXAMPLE 8  Finding the Optimal Angle of Elevation of a Shot Put

The optimal angle of elevation \( \theta \) a shot-putter should aim for to throw the greatest distance depends on the velocity \( v \) of the throw and the initial height \( h \) of the shot. See Figure 25. One model for \( \theta \) that achieves this greatest distance is

\[
\theta = \arcsin\left(\frac{v}{\sqrt{2v^2 + 64h}}\right).
\]

(Source: Townend, M. Stewart, Mathematics in Sport, Chichester, Ellis Horwood Limited, 1984.)

\[
\text{Figure 25}
\]

Suppose a shot-putter can consistently throw the steel ball with \( h = 6.6 \) ft and \( v = 42 \) ft per sec. At what angle should he throw the ball to maximize distance?

Solution  To find this angle, substitute and use a calculator in degree mode.

\[
\theta = \arcsin\left(\frac{42^2}{\sqrt{2 \cdot 42^2 + 64(6.6)}}\right) \approx 41.9^\circ \quad h = 6.6, v = 42
\]

Now try Exercise 93.
7.5 Exercises

1. one-to-one  2. range  3. domain
4. \(y = \tan^{-1} x\) or \(y = \cot^{-1} x\) or \(y = \arctan x\)  5. \(\pi\)
6. Sketch the reflection of the graph of \(f\) across the line \(y = x\).
7. (a) \([-1, 1]\) (b) \([-\frac{\pi}{2}, \frac{\pi}{2}]\)
   (c) increasing  (d) \(-2\) is not in the domain.
8. (a) \([-1, 1]\) (b) \([0, \pi]\)
   (c) decreasing  (d) \(-\frac{4\pi}{3}\) is not in the range.
9. (a) \((-\infty, \infty)\) (b) \((-\frac{\pi}{2}, \frac{\pi}{2})\)
   (c) increasing  (d) no
10. (a) \((-\infty, -1] \cup [1, \infty);\)
    \([-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})\)
    (b) \((-\infty, -1] \cup [1, \infty);\)
    \([0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)\)
    (c) \((-\infty, \infty);\) \((0, \pi)\)
11. \(\cos^{-1} \frac{1}{a}\)  12. Find \(\tan^{-1} \frac{1}{a} + \pi\) (or 180°).
13. 0  14. \(\frac{\pi}{4}\)  15. \(\pi\)  16. \(-\frac{\pi}{4}\)
17. \(-\frac{\pi}{2}\)  18. \(\frac{\pi}{3}\)  19. 0
20. \(-\frac{\pi}{3}\)  21. \(\frac{\pi}{2}\)  22. \(-\frac{\pi}{4}\)
23. \(-\frac{\pi}{4}\)  24. \(\frac{2\pi}{3}\)  25. \(\frac{5\pi}{6}\)
26. \(\frac{3\pi}{4}\)  27. \(\frac{3\pi}{4}\)  28. \(\frac{3\pi}{4}\)
29. \(\frac{5\pi}{6}\)  30. \(\frac{5\pi}{6}\)  31. \(\pi\)
32. \(\frac{\pi}{4}\)  33. \(-45°\)  34. \(120°\)
35. \(-60°\)  36. \(-45°\)  37. \(120°\)

Concept Check  Complete each statement.
1. For a function to have an inverse, it must be ________.
2. The domain of \(y = \arcsin x\) equals the ________ of \(y = \sin x\).
3. The range of \(y = \cos^{-1} x\) equals the ________ of \(y = \cos x\).
4. The point \((\frac{\pi}{2}, 1)\) lies on the graph of \(y = \tan x\). Therefore, the point ________ lies on the graph of ________.
5. If a function \(f\) has an inverse and \(f(\pi) = -1\), then \(f^{-1}(-1) = ________\).
6. How can the graph of \(f^{-1}\) be sketched if the graph of \(f\) is known?

Concept Check  In Exercises 7–10, write short answers.
7. Consider the inverse sine function, defined by \(y = \sin^{-1} x\) or \(y = \arcsin x\).
   (a) What is its domain?  (b) What is its range?
   (c) Is this function increasing or decreasing?
   (d) Why is \(\arcsin(-2)\) not defined?
8. Consider the inverse cosine function, defined by \(y = \cos^{-1} x\) or \(y = \arccos x\).
   (a) What is its domain?  (b) What is its range?
   (c) Is this function increasing or decreasing?
   (d) Arccos \((-\frac{1}{2})\) = \(\frac{2\pi}{3}\). Why is \(\arccos(-\frac{1}{2})\) not equal to \(-\frac{4\pi}{3}\)?
9. Consider the inverse tangent function, defined by \(y = \tan^{-1} x\) or \(y = \arctan x\).
   (a) What is its domain?  (b) What is its range?
   (c) Is this function increasing or decreasing?
   (d) Is there any real number \(x\) for which \(\arctan x\) is not defined? If so, what is it (or what are they)?
10. Give the domain and range of the three other inverse trigonometric functions, as defined in this section.
   (a) inverse cosecant function  (b) inverse secant function
   (c) inverse cotangent function

Concept Check  Is \(\sec^{-1} a\) calculated as \(\cos^{-1} \frac{1}{a}\) or as \(\frac{1}{\cos^{-1} a}\) ?

12. Concept Check  For positive values of \(a\), \(\cot^{-1} a\) is calculated as \(\tan^{-1} \frac{1}{a}\). How is \(\cot^{-1} a\) calculated for negative values of \(a\)?

Find the exact value of each real number \(y\). Do not use a calculator. See Examples 1 and 2.
13. \(y = \sin^{-1} 0\)  14. \(y = \tan^{-1} 1\)  15. \(y = \cos^{-1}(-1)\)
16. \(y = \arctan(-1)\)  17. \(y = \sin^{-1}(-1)\)  18. \(y = \cos^{-1} \frac{1}{2}\)
19. \(y = \arctan 0\)  20. \(y = \arcsin \left(-\frac{\sqrt{3}}{2}\right)\)  21. \(y = \arccos 0\)
22. \(y = \tan^{-1}(-1)\)  23. \(y = \sin^{-1} \frac{\sqrt{3}}{2}\)  24. \(y = \cos^{-1}(-\frac{1}{2})\)
25. \(y = \arccos \left(-\frac{\sqrt{3}}{2}\right)\)  26. \(y = \arcsin \left(-\frac{\sqrt{3}}{2}\right)\)  27. \(y = \cot^{-1}(-1)\)
28. \(y = \sec^{-1}(-\sqrt{2})\)  29. \(y = \csc^{-1}(-2)\)  30. \(y = \arccsc(-\sqrt{3})\)
31. \(y = \arccsc \frac{2\sqrt{3}}{3}\)  32. \(y = \csc^{-1} \sqrt{2}\)
38. $-30^\circ$  39. $120^\circ$  40. $-90^\circ$
41. $-7.6713835^\circ$
42. $97.671207^\circ$
43. $113.500970^\circ$
44. $51.1691219^\circ$
45. $30.987961^\circ$  46. $29.506181^\circ$
47. $83798122^\circ$  48. $96012698^\circ$
49. $2.3154725$  50. $2.4605221$
51. $1.1900238$  52. $1.003$

Give the degree measure of $\theta$. Do not use a calculator. See Example 3.

33. $\theta = \arctan(-1)$  34. $\theta = \arccos\left(-\frac{1}{2}\right)$  35. $\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$
36. $\theta = \arcsin\left(-\frac{\sqrt{2}}{2}\right)$  37. $\theta = \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$  38. $\theta = \csc^{-1}(-2)$
39. $\theta = \sec^{-1}(-2)$  40. $\theta = \csc^{-1}(-1)$

Use a calculator to give each value in decimal degrees. See Example 4.

41. $\theta = \sin^{-1}(-.13349122)$  42. $\theta = \cos^{-1}(-.13348816)$
43. $\theta = \arccos(-.39876459)$  44. $\theta = \arcsin .77900016$
45. $\theta = \csc^{-1}1.9422833$  46. $\theta = \cot^{-1}1.7670492$

Use a calculator to give each real number value. (Be sure the calculator is in radian mode.) See Example 4.

47. $y = \arctan 1.1111111$  48. $y = \arcsin .81926439$
49. $y = \cot^{-1}(-.92170128)$  50. $y = \sec^{-1}(-1.2871684)$
51. $y = \arcsin .92837781$  52. $y = \arccos .44624593$

Graph each inverse function as defined in the text.

53. $y = \cot^{-1}x$  54. $y = \csc^{-1}x$  55. $y = \sec^{-1}x$
56. $y = \csc 2x$  57. $y = \arccot \frac{1}{2}x$

58. 1.003 is not in the domain of $y = \sin^{-1}x$.
59. The domain of $y = \tan^{-1}x$ is $(-\infty, \infty)$.
60. In both cases, the result is $x$. In each case, the graph is a straight line bisecting quadrants I and III (i.e., the line $y = x$).
61. It is the graph of $y = x$.

62. It does not agree because the range of the inverse tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, not $(-\infty, \infty)$, as discussed in Exercise 61.

63. 64. 65.

Relating Concepts

For individual or collaborative investigation
(Exercises 60–62)*

60. Consider the function defined by $f(x) = 3x - 2$ and its inverse $f^{-1}(x) = \frac{x + 2}{3}$. Simplify $f[f^{-1}(x)]$ and $f^{-1}[f(x)]$. What do you notice in each case? What would the graph look like in each case?

61. Use a graphing calculator to graph $y = \tan(\tan^{-1}x)$ in the standard viewing window, using radian mode. How does this compare to the graph you described in Exercise 60?

62. Use a graphing calculator to graph $y = \tan^{-1}\tan x$ in the standard viewing window, using radian and dot modes. Why does this graph not agree with the graph you found in Exercise 61?

*The authors wish to thank Carol Walker of Hinds Community College for making a suggestion on which these exercises are based.
Give the exact value of each expression without using a calculator. See Examples 5 and 6.

63. \( \tan \left( \arccos \frac{3}{4} \right) \)  
64. \( \sin \left( \arccos \frac{1}{4} \right) \)  
65. \( \cos(\tan^{-1}(-2)) \)

66. \( \sec(\sin^{-1}\left(-\frac{1}{5}\right)) \)  
67. \( \sin(2 \tan^{-1} \frac{12}{5}) \)  
68. \( \cos\left(2 \sin^{-1} \frac{1}{4}\right) \)

69. \( \cos\left(2 \tan^{-1} \frac{4}{3}\right) \)  
70. \( \tan\left(2 \cos^{-1} \frac{1}{4}\right) \)

71. \( \sin\left(2 \cos^{-1} \frac{1}{5}\right) \)

72. \( \cos(2 \tan^{-1}(-2)) \)  
73. \( \sec(\sec^{-1} 2) \)  
74. \( \csc(\csc^{-1} \sqrt{2}) \)

75. \( \cos\left(\tan^{-1} \frac{5}{12} - \tan^{-1} \frac{3}{4}\right) \)  
76. \( \cos\left(\sin^{-1} \frac{3}{5} + \cos^{-1} \frac{5}{13}\right) \)

77. \( \sin\left(\sin^{-1} \frac{1}{2} + \tan^{-1}(-3)\right) \)  
78. \( \tan\left(\cos^{-1} \frac{\sqrt{3}}{2} - \sin^{-1} \left(-\frac{3}{5}\right)\right) \)

Use a calculator to find each value. Give answers as real numbers.

79. \( \cos(\tan^{-1} .5) \)  
80. \( \sin(\cos^{-1} .25) \)

81. \( \tan(\arcsin .12251014) \)  
82. \( \cot(\arccos .58236841) \)

Write each expression as an algebraic (nontrigonometric) expression in \( u, u > 0 \). See Example 7.

83. \( \sin(\arccos u) \)  
84. \( \tan(\arccos u) \)  
85. \( \cos(\arcsin u) \)

86. \( \cot(\arcsin u) \)  
87. \( \sin\left(\sec^{-1} \frac{u}{2}\right) \)  
88. \( \cos\left(\tan^{-1} \frac{3}{u}\right) \)

89. \( \tan\left(\sin^{-1} \frac{u}{\sqrt{u^2 + 2}}\right) \)  
90. \( \sec\left(\cos^{-1} \frac{u}{\sqrt{u^2 + 5}}\right) \)

91. \( \sec\left(\arccot \frac{\sqrt{4 - u^2}}{u}\right) \)  
92. \( \csc\left(\arctan \frac{\sqrt{9 - u^2}}{u}\right) \)

(Modeling) Solve each problem.

93. Angle of Elevation of a Shot Put Refer to Example 8.

(a) What is the optimal angle when \( h = 0? \)

(b) Fix \( h \) at 6 ft and regard \( \theta \) as a function of \( v \). As \( v \) gets larger and larger, the graph approaches an asymptote. Find the equation of that asymptote.

94. Angle of Elevation of a Plane Suppose an airplane flying faster than sound goes directly over you. Assume that the plane is flying at a constant altitude. At the instant you feel the sonic boom from the plane, the angle of elevation to the plane is given by

\[ \theta = 2 \arcsin \frac{1}{m}, \]

where \( m \) is the Mach number of the plane’s speed. (The Mach number is the ratio of the speed of the plane and the speed of sound.) Find \( \theta \) to the nearest degree for each value of \( m. \)

(a) \( m = 1.2 \)  
(b) \( m = 1.5 \)  
(c) \( m = 2 \)  
(d) \( m = 2.5 \)
95. **Observation of a Painting**  A painting 1 m high and 3 m from the floor will cut off an angle \( \theta \) to an observer, where

\[
\theta = \tan^{-1}\left( \frac{x}{x^2 + 2} \right).
\]

Assume that the observer is \( x \) meters from the wall where the painting is displayed and that the eyes of the observer are 2 m above the ground. (See the figure.) Find the value of \( \theta \) for the following values of \( x \). Round to the nearest degree.

(a) 1  
(b) 2  
(c) 3  
(d) Derive the formula given above. (*Hint:* Use the identity for \( \tan(\theta + \alpha) \). Use right triangles.)

(e) Graph the function for \( \theta \) with a graphing calculator, and determine the distance that maximizes the angle.

(f) The idea in part (e) was first investigated in 1471 by the astronomer Regiomontanus. (*Source:* Maor, E., *Trigonometric Delights*, Princeton University Press, 1998.) If the bottom of the picture is \( a \) meters above eye level and the top of the picture is \( b \) meters above eye level, then the optimum value of \( x \) is \( \sqrt{ab} \) meters. Use this result to find the exact answer to part (e).

96. **Landscaping Formula**  A shrub is planted in a 100-ft-wide space between buildings measuring 75 ft and 150 ft tall. The location of the shrub determines how much sun it receives each day. Show that if \( \theta \) is the angle in the figure and \( x \) is the distance of the shrub from the taller building, then the value of \( \theta \) (in radians) is given by

\[
\theta = \pi - \arctan\left( \frac{75}{100 - x} \right) - \arctan\left( \frac{150}{x} \right).
\]

97. **Communications Satellite Coverage**  The figure shows a stationary communications satellite positioned 20,000 mi above the equator. What percent of the equator can be seen from the satellite? The diameter of Earth is 7927 mi at the equator.
7.6 Trigonometric Equations

Looking Ahead to Calculus

There are many instances in calculus where it is necessary to solve trigonometric equations. Examples include solving related-rates problems and optimization problems.

Earlier in this chapter, we studied trigonometric equations that were identities. We now consider trigonometric equations that are conditional; that is, equations that are satisfied by some values but not others.

Solving by Linear Methods

Conditional equations with trigonometric (or circular) functions can usually be solved using algebraic methods and trigonometric identities.

EXAMPLE 1 Solving a Trigonometric Equation by Linear Methods

Solve \(2 \sin \theta - 1 = 0\) over the interval \([0^\circ, 360^\circ]\).

**Solution** Because \(\sin \theta\) is the first power of a trigonometric function, we use the same method as we would to solve the linear equation \(2x + 1 = 0\).

\[
\begin{align*}
2 \sin \theta + 1 &= 0 \\
2 \sin \theta &= -1 \\
\sin \theta &= -\frac{1}{2}
\end{align*}
\]

To find values of \(\theta\) that satisfy \(\sin \theta = -\frac{1}{2}\), we observe that \(\theta\) must be in either quadrant III or IV since the sine function is negative only in these two quadrants. Furthermore, the reference angle must be \(30^\circ\) since \(\sin 30^\circ = \frac{1}{2}\). The graphs in Figure 26 show the two possible values of \(\theta\), \(210^\circ\) and \(330^\circ\). The solution set is \(\{210^\circ, 330^\circ\}\).

Alternatively, we could determine the solutions by referring to Figure 11 in Section 6.2 on page 546.

Now try Exercise 11.

Solving by Factoring

EXAMPLE 2 Solving a Trigonometric Equation by Factoring

Solve \(\sin x \tan x = \sin x\) over the interval \([0^\circ, 360^\circ]\).

**Solution**

\[
\begin{align*}
\sin x \tan x &= \sin x \\
\sin x \tan x - \sin x &= 0 \\
\sin x (\tan x - 1) &= 0 \\
\sin x &= 0 \quad \text{or} \quad \tan x - 1 = 0 \\
\tan x &= 1
\end{align*}
\]

The solution set is \(\{0^\circ, 45^\circ, 180^\circ, 225^\circ\}\).

Now try Exercise 31.
CAUTION There are four solutions in Example 2. Trying to solve the equation by dividing each side by \( \sin x \) would lead to just which would give \( x = 45^\circ \) or \( x = 225^\circ \). The other two solutions would not appear. The missing solutions are the ones that make the divisor, \( \sin x \), equal 0. For this reason, we avoid dividing by a variable expression.

Solving by Quadratic Methods In Section 1.6, we saw that an equation in the form \( au^2 + bu + c = 0 \), where \( u \) is an algebraic expression, is solved by quadratic methods. The expression \( u \) may also be a trigonometric function, as in the equation \( \tan^2 x + \tan x - 2 = 0 \).

EXAMPLE 3 Solving a Trigonometric Equation by Factoring

Solve \( \tan^2 x + \tan x - 2 = 0 \) over the interval \([0, 2\pi)\).

**Solution** This equation is quadratic in form and can be solved by factoring.

\[
\begin{align*}
\tan^2 x + \tan x - 2 &= 0 \\
(tan x - 1)(\tan x + 2) &= 0 & \text{Factor.} \\
\tan x - 1 &= 0 & \text{or} & \tan x + 2 &= 0 & \text{Zero-factor property} \\
\tan x &= 1 & \text{or} & \tan x &= -2 \\
\end{align*}
\]

The solutions for \( \tan x = 1 \) over the interval \([0, 2\pi)\) are \( x = \frac{\pi}{4} \) and \( x = \frac{5\pi}{4} \).

To solve \( \tan x = -2 \) over that interval, we use a scientific calculator set in radian mode. We find that \( \tan^{-1}(-2) \approx -1.1071487 \). This is a quadrant IV number, based on the range of the inverse tangent function. (Refer to Figure 11 in Section 6.2 on page 546.) However, since we want solutions over the interval \([0, 2\pi)\), we must first add \( \pi \) to \(-1.1071487\), and then add \( 2\pi \).

\[
\begin{align*}
x &= -1.1071487 + \pi \approx 2.034439 \\
x &= -1.1071487 + 2\pi \approx 5.1760366 \\
\end{align*}
\]

The solutions over the required interval form the solution set

\[
\left\{ \frac{\pi}{4}, \frac{5\pi}{4}, 2.0, 5.2 \right\}.
\]

Now try Exercise 21.

EXAMPLE 4 Solving a Trigonometric Equation Using the Quadratic Formula

Find all solutions of \( \cot x(\cot x + 3) = 1 \).

**Solution** We multiply the factors on the left and subtract 1 to get the equation in standard quadratic form.

\[
\cot^2 x + 3 \cot x - 1 = 0 \quad \text{(Section 1.4)}
\]

Since this equation cannot be solved by factoring, we use the quadratic formula, with \( a = 1 \), \( b = 3 \), \( c = -1 \), and \( \cot x \) as the variable.
660  CHAPTER 7  Trigonometric Identities and Equations

**Teaching Tip** Unlike the cosine function, equations of the form \( y = \tan x \) will not contain a second value for \( x \) between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \). Remind students that other solutions to \( \tan x = y \) are found using period \( \pi \) radians.

\[
\cot x = \frac{-3 \pm \sqrt{9 + 4}}{2} = \frac{-3 \pm \sqrt{13}}{2}
\]

We cannot find inverse cotangent values directly on a calculator, so we use the fact that \( \cot x = \frac{1}{\tan x} \), and take reciprocals to get
\[
\tan x = -0.302775638 \quad \text{or} \quad \tan x = 3.027756377
\]
\[
x = -0.2940013018 \quad \text{or} \quad x = 1.276795025.
\]

To find all solutions, we add integer multiples of the period of the tangent function, which is \( \pi \), to each solution found above. Thus, all solutions of the equation are written as
\[
-0.2940013018 + n\pi \quad \text{and} \quad 1.276795025 + n\pi, \quad \text{where} \ n \ \text{is any integer}.*
\]

Now try Exercise 43.

### Solving by Using Trigonometric Identities

Recall that squaring both sides of an equation, such as \( \sqrt{x + 4} = x + 2 \), will yield all solutions but may also give extraneous values. (In this equation, 0 is a solution, while \(-3\) is extraneous. Verify this.) The same situation may occur when trigonometric equations are solved in this manner.

**Example 5**  Solving a Trigonometric Equation by Squaring

Solve \( \tan x + \sqrt{3} = \sec x \) over the interval \([0, 2\pi)\).

**Solution**  Since the tangent and secant functions are related by the identity \( 1 + \tan^2 x = \sec^2 x \), square both sides and express \( \sec^2 x \) in terms of \( \tan^2 x \).

\[
\tan x + \sqrt{3} = \sec x
\]

\[
\tan^2 x + 2\sqrt{3} \tan x + 3 = \sec^2 x
\]

\[
(x + y)^2 = x^2 + 2xy + y^2 \quad \text{(Section R.3)}
\]

\[
\tan^2 x + 2 \sqrt{3} \tan x + 3 = 1 + \tan^2 x
\]

\[
2\sqrt{3} \tan x = -2
\]

\[
\tan x = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}
\]

\[
\text{Divide by} \ 2\sqrt{3}; \ \text{rationalize the denominator (Section R.7)}
\]

The possible solutions are \( \frac{5\pi}{6} \) and \( \frac{11\pi}{6} \). Now check them. Try \( \frac{5\pi}{6} \) first.

Left side:
\[
\tan x + \sqrt{3} = \tan \frac{5\pi}{6} + \sqrt{3} = -\frac{\sqrt{3}}{3} + \sqrt{3} = \frac{2\sqrt{3}}{3}
\]

Right side:
\[
\sec x = \sec \frac{5\pi}{6} = -\frac{2\sqrt{3}}{3}
\]

\( \text{Not equal} \)

*We usually give solutions of equations as solution sets, except when we ask for all solutions of a trigonometric equation.
The check shows that \(\frac{5\pi}{6}\) is not a solution. Now check \(\frac{11\pi}{6}\).

Left side: \[\tan \frac{11\pi}{6} + \sqrt{3} = -\frac{\sqrt{3}}{3} + \sqrt{3} = \frac{2\sqrt{3}}{3}\]

Right side: \[\sec \frac{11\pi}{6} = \frac{2\sqrt{3}}{3}\]

This solution satisfies the equation, so \(\left\{\frac{11\pi}{6}\right\}\) is the solution set.

Now try Exercise 41.

The methods for solving trigonometric equations illustrated in the examples can be summarized as follows.

**Solving a Trigonometric Equation**

1. Decide whether the equation is linear or quadratic in form, so you can determine the solution method.
2. If only one trigonometric function is present, first solve the equation for that function.
3. If more than one trigonometric function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to 0 to solve.
4. If the equation is quadratic in form, but not factorable, use the quadratic formula. Check that solutions are in the desired interval.
5. Try using identities to change the form of the equation. It may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions.

Some trigonometric equations involve functions of half-angles or multiple angles.

**Equations with Half-Angles**

**EXAMPLE 6  Solving an Equation Using a Half-Angle Identity**

Solve \(2 \sin \frac{x}{2} = 1\)

(a) over the interval \([0, 2\pi]\), and (b) give all solutions.

**Solution**

(a) Write the interval \([0, 2\pi]\) as the inequality

\[0 \leq x < 2\pi.\]

The corresponding interval for \(\frac{x}{2}\) is

\[0 \leq \frac{x}{2} < \pi. \quad \text{Divide by 2. (Section 1.7)}\]
To find all values of $\frac{x}{2}$ over the interval $[0, \pi)$ that satisfy the given equation, first solve for $\sin \frac{x}{2}$.

$$2 \sin \frac{x}{2} = 1$$
$$\sin \frac{x}{2} = \frac{1}{2}$$

Divide by 2.

The two numbers over the interval $[0, \pi]$ with sine value $\frac{1}{2}$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, so

$$\frac{x}{2} = \frac{\pi}{6} \quad \text{or} \quad \frac{x}{2} = \frac{5\pi}{6}$$

$$x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}$$

Multiply by 2.

The solution set over the given interval is $\left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\}$.

(b) Since this is a sine function with period $4\pi$, all solutions are given by the expressions

$$\frac{\pi}{3} + 4n\pi \quad \text{and} \quad \frac{5\pi}{3} + 4n\pi, \quad \text{where } n \text{ is any integer.}$$

Now try Exercise 63.

### Equations with Multiple Angles

#### EXAMPLE 7 Solving an Equation with a Double Angle

Solve $\cos 2x = \cos x$ over the interval $[0, 2\pi)$.

**Solution** First change $\cos 2x$ to a trigonometric function of $x$. Use the identity $\cos 2x = 2\cos^2 x - 1$ so the equation involves only $\cos x$. Then factor.

$$\cos 2x = \cos x$$
$$2\cos^2 x - 1 = \cos x$$

Substitute; double-angle identity (Section 7.4)

$$2\cos^2 x - \cos x - 1 = 0$$

Subtract $\cos x$.

$$(2\cos x + 1)(\cos x - 1) = 0$$

Factor.

$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0$$

Zero-factor property

$$\cos x = -\frac{1}{2} \quad \text{or} \quad \cos x = 1$$

Over the required interval,

$$x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3} \quad \text{or} \quad x = 0.$$ 

The solution set is $\left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\}$.

Now try Exercise 65.
In the solution of Example 7, \( \cos 2x \) cannot be changed to \( \cos x \) by dividing by 2 since 2 is not a factor of \( \cos 2x \).

\[
\frac{\cos 2x}{2} \neq \cos x
\]

The only way to change \( \cos 2x \) to a trigonometric function of \( x \) is by using one of the identities for \( \cos 2x \).

**EXAMPLE 8**  Solving an Equation Using a Multiple-Angle Identity

Solve \( 4 \sin \theta \cos \theta = \sqrt{3} \) over the interval \([0^\circ, 360^\circ]\).

**Solution**  The identity \( 2 \sin \theta \cos \theta = \sin 2\theta \) is useful here.

\[
\begin{align*}
4 \sin \theta \cos \theta &= \sqrt{3} \\
2(2 \sin \theta \cos \theta) &= \sqrt{3} \\
4 &= 2 \cdot 2 \\
2 \sin 2\theta &= \sqrt{3} \\
\sin 2\theta &= \frac{\sqrt{3}}{2}
\end{align*}
\]

Divide by 2.

From the given interval \( 0^\circ \leq \theta < 360^\circ \), the interval for \( 2\theta \) is \( 0^\circ \leq 2\theta < 720^\circ \). List all solutions over this interval.

\[
2\theta = 60^\circ, 120^\circ, 420^\circ, 480^\circ
\]

or

\[
\theta = 30^\circ, 60^\circ, 210^\circ, 240^\circ
\]

The final two solutions for \( 2\theta \) were found by adding 360\(^\circ\) to 60\(^\circ\) and 120\(^\circ\), respectively, giving the solution set \(\{30^\circ, 60^\circ, 210^\circ, 240^\circ\}\).

Now try Exercise 83.

**Applications**  Music is closely related to mathematics.

**EXAMPLE 9**  Describing a Musical Tone from a Graph

A basic component of music is a pure tone. The graph in Figure 27 models the sinusoidal pressure \( y = P \) in pounds per square foot from a pure tone at time \( x = t \) in seconds.

(a) The frequency of a pure tone is often measured in hertz. One hertz is equal to one cycle per second and is abbreviated Hz. What is the frequency \( f \) in hertz of the pure tone shown in the graph?

(b) The time for the tone to produce one complete cycle is called the *period*. Approximate the period \( T \) in seconds of the pure tone.

(c) An equation for the graph is \( y = \cdot04 \sin(300\pi x) \). Use a calculator to estimate all solutions to the equation that make \( y = .004 \) over the interval \([0, .02]\).
Solution

(a) From the graph in Figure 27 on the previous page, we see that there are 6 cycles in .04 sec. This is equivalent to \( \frac{6}{.04} = 150 \) cycles per sec. The pure tone has a frequency of \( f = 150 \) Hz.

(b) Six periods cover a time of .04 sec. One period would be equal to \( T = \frac{.04}{6} = \frac{1}{150} \) or .006 sec.

(c) If we reproduce the graph in Figure 27 on a calculator as \( Y_1 \) and also graph a second function as \( Y_2 = .004 \), we can determine that the approximate values of \( x \) at the points of intersection of the graphs over the interval are 

\[ .0017, .0083, \text{ and } .015. \]

The first value is shown in Figure 28.

Now try Exercise 87.

A piano string can vibrate at more than one frequency when it is struck. It produces a complex wave that can mathematically be modeled by a sum of several pure tones. If a piano key with a frequency of \( f_1 \) is played, then the corresponding string will not only vibrate at \( f_1 \) but it will also vibrate at the higher frequencies of \( 2f_1, 3f_1, 4f_1, \ldots, nf_1, \ldots, f_1 \) is called the fundamental frequency of the string, and higher frequencies are called the upper harmonics. The human ear will hear the sum of these frequencies as one complex tone. (Source: Roederer, J., Introduction to the Physics and Psychophysics of Music, Second Edition, Springer-Verlag, 1975.)

**EXAMPLE 10 Analyzing Pressures of Upper Harmonics**

Suppose that the A key above middle C is played. Its fundamental frequency is \( f_1 = 440 \) Hz, and its associated pressure is expressed as

\[ P_1 = .002 \sin(880 \pi t). \]

The string will also vibrate at

\[ f_2 = 880, \quad f_3 = 1320, \quad f_4 = 1760, \quad f_5 = 2200, \ldots \] Hz.

The corresponding pressures of these upper harmonics are

\[ P_2 = \frac{.002}{2} \sin(1760 \pi t), \quad P_3 = \frac{.002}{3} \sin(2640 \pi t), \]

\[ P_4 = \frac{.002}{4} \sin(3520 \pi t), \quad \text{and} \quad P_5 = \frac{.002}{5} \sin(4400 \pi t). \]

The graph of

\[ P = P_1 + P_2 + P_3 + P_4 + P_5, \]

shown in Figure 29, is “saw-toothed.”

(a) What is the maximum value of \( P \)?

(b) At what values of \( x \) does this maximum occur over the interval \([0, .01]\)?
### 7.6 Trigonometric Equations

#### Solution

(a) A graphing calculator shows that the maximum value of $P$ is approximately 0.00317. See Figure 30.

(b) The maximum occurs at $x = 0.000188, 0.00246, 0.00474, 0.00701,$ and 0.00928. Figure 30 shows how the second value is found; the others are found similarly.

Now try Exercise 89.

1. Solve the linear equation for $\cot x$.
2. Solve the linear equation for $\sin x$.
3. Solve the quadratic equation for $\sec x$ by factoring.
4. Solve the quadratic equation for $\cos x$ by the zero-factor property.
5. Solve the quadratic equation for $\sin x$ using the quadratic formula.
6. Solve the quadratic equation for $\tan x$ using the quadratic formula.
7. Use an identity to rewrite as an equation with one trigonometric function.

#### Concept Check
Refer to the summary box on solving a trigonometric equation. Decide on the appropriate technique to begin the solution of each equation. Do not solve the equation.

1. $2 \cot x + 1 = -1$
2. $\sin x + 2 = 3$
3. $5 \sec^2 x = 6 \sec x$
4. $2 \cos^2 x - \cos x = 1$
5. $9 \sin^2 x - 5 \sin x = 1$
6. $\tan^2 x - 4 \tan x + 2 = 0$
7. $\tan x - \cot x = 0$
8. $\cos^2 x = \sin^2 x + 1$

#### Concept Check
Answer each question.

9. Suppose you are solving a trigonometric equation for solutions over the interval $[0, 2\pi)$, and your work leads to $2x = \frac{2\pi}{3}, 2\pi, \frac{5\pi}{3}$. What are the corresponding values of $x$?

10. Suppose you are solving a trigonometric equation for solutions over the interval $[0^\circ, 360^\circ)$, and your work leads to $\frac{1}{2} \theta = 45^\circ, 60^\circ, 75^\circ, 90^\circ$. What are the corresponding values of $\theta$?

#### Exercises

8. Use an identity to rewrite as an equation with one trigonometric function.

9. $\frac{\pi}{3}, \pi, \frac{4\pi}{3}$
10. $135^\circ, 180^\circ$.
11. $225^\circ, 270^\circ$
12. $\left\{ \frac{\pi}{2} \right\}$
13. $\left\{ \frac{\pi}{5}, \frac{7\pi}{6} \right\}$
14. $\left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\}$
15. $\emptyset$
16. $\emptyset$
17. $\left\{ \frac{\pi}{4}, \frac{5\pi}{4}, \frac{5\pi}{6} \right\}$
18. $\left\{ \frac{\pi}{4}, \frac{7\pi}{6}, \frac{11\pi}{6} \right\}$
19. $\{ \pi \}$
20. $\left\{ \frac{\pi}{2}, \frac{3\pi}{2}, \frac{11\pi}{6} \right\}$
21. $\left\{ \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \right\}$
22. $\left\{ 0, \frac{4\pi}{3}, \frac{2\pi}{3} \right\}$
23. $\{ 30^\circ, 210^\circ, 240^\circ, 300^\circ \}$
24. $\{ 0^\circ, 45^\circ, 225^\circ \}$
25. $\{ 90^\circ, 210^\circ, 330^\circ \}$
26. $\{ 60^\circ, 135^\circ, 240^\circ, 315^\circ \}$
27. $\{ 45^\circ, 135^\circ, 225^\circ, 315^\circ \}$
28. $\{ 0^\circ, 180^\circ \}$
29. $\{ 45^\circ, 225^\circ \}$
30. $\{ 90^\circ, 270^\circ \}$
31. $\{ 0^\circ, 30^\circ, 150^\circ, 180^\circ \}$
32. $\{ 0^\circ, 90^\circ, 180^\circ, 270^\circ \}$
33. $\{ 0^\circ, 45^\circ, 135^\circ, 180^\circ, 225^\circ, 315^\circ \}$
34. $\{ 45^\circ, 135^\circ, 225^\circ, 315^\circ \}$
35. $\{ 53.6^\circ, 126.4^\circ, 187.9^\circ, 352.1^\circ \}$
36. $\{ 78.0^\circ, 282.0^\circ \}$

#### Solve each equation for exact solutions over the interval $[0, 2\pi)$.

11. $2 \cot x + 1 = -1$
12. $\sin x + 2 = 3$
13. $2 \sin x + 3 = 4$
14. $2 \sec x + 1 = \sec x + 3$
15. $\tan^2 x + 3 = 0$
16. $\sec^2 x + 2 = -1$
17. $(\cot x - 1)(\sqrt{3} \cot x + 1) = 0$
18. $(\csc x + 2)(\csc x - \sqrt{3}) = 0$
19. $\cos^2 x + 2 \cos x + 1 = 0$
20. $2 \cos^2 x - \sqrt{3} \cos x = 0$
21. $-2 \sin^2 x = 3 \sin x + 1$
22. $2 \cos^2 x - \cos x = 1$

#### Solve each equation for exact solutions over the interval $[0^\circ, 360^\circ)$.

23. $(\cot \theta - \sqrt{3})(2 \sin \theta + \sqrt{3}) = 0$
24. $(\tan \theta - 1)(\cos \theta - 1) = 0$
25. $2 \sin \theta - 1 = \csc \theta$
26. $\tan \theta + 1 = \sqrt{3} + \sqrt{3} \cot \theta$
27. $\tan \theta - \cot \theta = 0$
28. $\cos^2 \theta = \sin^2 \theta + 1$
29. $\csc^2 \theta - 2 \cot \theta = 0$
30. $\sin^2 \theta \cos \theta = \cos \theta$
31. $2 \tan^2 \theta \sin \theta - \tan^2 \theta = 0$
32. $\sin^2 \theta \cos^2 \theta = 0$
33. $\sec^2 \theta \tan \theta = 2 \tan \theta$
34. $\cos^2 \theta - \sin^2 \theta = 0$
35. $9 \sin^2 \theta - 6 \sin \theta = 1$
36. $4 \cos^2 \theta + 4 \cos \theta = 1$
37. \{149.6^\circ, 329.6^\circ, 106.3^\circ, 286.3^\circ\}
38. \{38.4^\circ, 218.4^\circ, 104.8^\circ, 284.8^\circ\}
39. \theta \quad 40. \{68.5^\circ, 291.5^\circ\}
41. \{57.7^\circ, 159.2^\circ\}
42. \{114.3^\circ, 335.7^\circ\}
43. .9 + 2\pi n, 2.3 + 2\pi n, 3.6 + 2\pi n, 5.8 + 2\pi n, where n is any integer
44. \frac{\pi}{3} + 2\pi n, \pi + 2\pi n, \frac{5\pi}{3} + 2\pi n, where n is any integer
45. \frac{\pi}{3} + 2\pi n, \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n, where n is any integer
46. \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n, where n is any integer
47. 33.6^\circ + 360^\circ n, 326.4^\circ + 360^\circ n, where n is any integer
48. 90^\circ + 360^\circ n, 221.8^\circ + 360^\circ n, 318.2^\circ + 360^\circ n, where n is any integer
49. 45^\circ + 180^\circ n, 108.4^\circ + 180^\circ n, where n is an integer
50. 135^\circ + 360^\circ n, 315^\circ + 360^\circ n, 71.6^\circ + 360^\circ n, 251.6^\circ + 360^\circ n, where n is any integer
51. \{.6806, 1.4159\}
52. \{0, .3760\}
53. \left\{x \mid x \in \mathbb{R}, x \neq \frac{\pi}{4}, \frac{5\pi}{4}\right\}
54. \left\{x \mid x \in \mathbb{R}, x \neq \frac{\pi}{4}, \frac{5\pi}{4}\right\}
55. tan^2 \theta + 4 tan \theta + 2 = 0
56. sin^2 \theta - 2 sin \theta + 3 = 0
57. cot^2 \theta - 3 cot \theta - 1 = 0
58. 2 cos^2 \theta + 2 cos \theta - 1 = 0
59. 2 sin \theta = 1 - 2 cos \theta

Determine all solutions of each equation in radians (for x) or degrees (for \theta) to the nearest tenth as appropriate. See Example 4.

43. 3 sin^2 x - sin x - 1 = 0
44. 2 cos^2 x + cos x = 1
45. 4 cos^2 x - 1 = 0
46. 2 cos^2 x + 5 cos x + 2 = 0
47. 5 sec^2 \theta = 6 sec \theta
48. 2 tan \theta - 3 - tan^2 \theta = 1
49. \frac{2 tan \theta}{3 - tan^2 \theta} = 1
50. sec^2 \theta = 2 tan \theta + 4

The following equations cannot be solved by algebraic methods. Use a graphing calculator to find all solutions over the interval \([0, 2\pi)\). Express solutions to four decimal places.

51. \chi^2 + sin x - \chi^3 - cos x = 0
52. \chi^3 - cos^2 x = \frac{1}{2} \chi - 1

53. Explain what is wrong with the following solution.
Solve tan 2\theta = 2 over the interval \([0, 2\pi)\).
\begin{align*}
\tan 2\theta &= 2 \\
\frac{2 \tan \theta}{1 - \tan^2 \theta} &= 2 \\
\tan \theta &= 1 \\
\theta &= \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{5\pi}{4}
\end{align*}

The solutions are \frac{\pi}{4} and \frac{5\pi}{4}.

54. The equation cot \frac{\chi}{2} - \csc \frac{\chi}{2} - 1 = 0 has no solution over the interval \([0, 2\pi)\). Using this information, what can be said about the graph of
\[ y = \cot \frac{x}{2} - \csc \frac{x}{2} - 1 \]
over this interval? Confirm your answer by actually graphing the function over the interval.

Solve each equation for exact solutions over the interval \([0, 2\pi)\). See Examples 6–8.

55. cos 2x = \frac{\sqrt{3}}{2}
56. cos 2x = -\frac{1}{2}
57. sin 3x = -1
58. sin 3x = 0
59. 3 tan 3x = \sqrt{3}
60. cot 3x = \sqrt{3}
61. \sqrt{2} \cos 2x = -1
62. 2\sqrt{3} \sin 2x = \sqrt{3}
63. \sin \frac{x}{2} = \sqrt{2} - \sin \frac{x}{2}
64. tan 4x = 0
65. sin x = sin 2x
66. cos 2x - cos x = 0
67. 8 sec^2 \frac{x}{2} = 4
68. sin^2 \frac{x}{2} - 2 = 0
69. sin \frac{x}{2} = cos \frac{x}{2}
70. sec \frac{x}{2} = cos \frac{x}{2}
67. \[ \{15^\circ, 45^\circ, 135^\circ, 195^\circ, 225^\circ, 285^\circ\} \]

68. \[ \{0^\circ, 30^\circ, 150^\circ, 270^\circ\} \]

69. \[ \{120^\circ, 240^\circ\} \quad 70. \{0\} \]

70. Solve each equation in Exercises 71–78 for exact solutions over the interval \([0^\circ, 360^\circ]\).

In Exercises 79–86, give all exact solutions. See Examples 6–8.

71. \[ \sqrt{2} \sin 3\theta - 1 = 0 \]
72. \[ -2 \cos 2\theta = \sqrt{3} \]
73. \[ \cos \frac{\theta}{2} = 1 \]
74. \[ \sin \frac{\theta}{2} = 1 \]
75. \[ 2\sqrt{3} \sin \frac{\theta}{2} = 3 \]
76. \[ 2\sqrt{3} \cos \frac{\theta}{2} = -3 \]
77. \[ 2 \sin \theta = 2 \cos 2\theta \]
78. \[ \cos \theta - 1 = \cos 2\theta \]
79. \[ 1 - \sin \theta = \cos 2\theta \]
80. \[ \sin 2\theta = 2 \cos^2 \theta \]
81. \[ \csc \frac{\theta}{2} = 2 \sec \theta \]
82. \[ \cos \theta = \sin^2 \frac{\theta}{2} \]
83. \[ 2 - \sin 2\theta = 4 \sin 2\theta \]
84. \[ 4 \cos 2\theta = 8 \sin \theta \cos \theta \]
85. \[ 2 \cos^2 2\theta = 1 - \cos 2\theta \]
86. \[ \sin \theta - \sin 2\theta = 0 \]

(Modeling) Solve each problem. See Examples 9 and 10.

87. Pressure on the Eardrum

No musical instrument can generate a true pure tone. A pure tone has a unique, constant frequency and amplitude that sounds dull and uninteresting. The pressures caused by pure tones on the eardrum are sinusoidal. The change in pressure \(P\) in pounds per square foot on a person’s eardrum from a pure tone at time \(t\) in seconds can be modeled using the equation

\[ P = A \sin (2\pi f t + \phi), \]

where \(f\) is the frequency in cycles per second, and \(\phi\) is the phase angle. When \(P\) is positive, there is an increase in pressure and the eardrum is pushed inward; when \(P\) is negative, there is a decrease in pressure and the eardrum is pushed outward. (Source: Roederer, J., Introduction to the Physics and Psychophysics of Music, Second Edition, Springer-Verlag, 1975.) A graph of the tone middle C is shown in the figure.

(a) Determine algebraically the values of \(t\) for which \(P = 0\) over \([0, .005]\).

(b) From the graph and your answer in part (a), determine the interval for which \(P < 0\) over \([0, .005]\).

(c) Would an eardrum hearing this tone be vibrating outward or inward when \(P < 0\)?

88. Hearing Beats in Music

Musicians sometimes tune instruments by playing the same tone on two different instruments and listening for a phenomenon known as beats. Beats occur when two tones vary in frequency by only a few hertz. When the two instruments are in tune, the beats disappear. The ear hears beats because the pressure slowly rises and falls as a result of this slight variation in the frequency. This phenomenon can be seen using a graphing calculator. (Source: Pierce, J., The Science of Musical Sound, Scientific American Books, 1992.)

(a) Consider two tones with frequencies of 220 and 223 Hz and pressures

\[ P_1 = .005 \sin 440\pi t \quad \text{and} \quad P_2 = .005 \sin 446\pi t, \]

respectively. Graph the pressure

\[ P = P_1 + P_2 \]

felt by an eardrum over the 1-sec interval \([.15, 1.15]\). How many beats are there in 1 sec?

(b) Repeat part (a) with frequencies of 220 and 216 Hz.

(c) Determine a simple way to find the number of beats per second if the frequency of each tone is given.
88. (b) 4 beats per sec
For \( x = t \),
\[
P(t) = .005 \sin 440\pi t + .005 \sin 32\pi t
\]
(c) The number of beats is equal to the absolute value of the difference in the frequencies of the two tones.

89. (a) the two tones.
Difference in the frequencies of \( \frac{4}{3} \) the absolute value of the
(c) The number of beats is equal to the absolute value of the difference in the frequencies of the two tones.

89. Pressure of a Plucked String
If a string with a fundamental frequency of 110 Hz is plucked in the middle, it will vibrate at the odd harmonics of 110, 330, 550, \ldots Hz but not at the even harmonics of 220, 440, 660, \ldots Hz. The resulting pressure \( P \) caused by the string can be modeled by the equation
\[
P = .003 \sin 220\pi t + \frac{.003}{3} \sin 660\pi t + \frac{.003}{5} \sin 1100\pi t + \frac{.003}{7} \sin 1540\pi t.
\]

(a) Graph \( P \) in the window \([0, .03]\) by \([-0.05, 0.05]\).
(b) Use the graph to describe the shape of the sound wave that is produced.
(c) See Exercise 87. At lower frequencies, the inner ear will hear a tone only when the eardrum is moving outward. Determine the times over the interval \([0, .03]\) when this will occur.

90. Hearing Difference Tones
Small speakers like those found in older radios and telephones often cannot vibrate slower than 200 Hz—yet 35 keys on a piano have frequencies below 200 Hz. When a musical instrument creates a tone of 110 Hz, it also creates tones at 220, 330, 440, 550, 660, \ldots Hz. A small speaker cannot reproduce the 110-Hz vibration but it can reproduce the higher frequencies, which are called the upper harmonics. The low tones can still be heard because the speaker produces difference tones of the upper harmonics. The difference between consecutive frequencies is 110 Hz, and this difference tone will be heard by a listener. We can model this phenomenon using a graphing calculator. (Source: Benade, A., *Fundamentals of Musical Acoustics*, Dover Publications, 1990.)

(a) In the window \([0, .03]\) by \([-1, 1]\), graph the upper harmonics represented by the pressure
\[
P = \frac{1}{2} \sin[2\pi(220)t] + \frac{1}{3} \sin[2\pi(330)t] + \frac{1}{4} \sin[2\pi(440)t].
\]
(b) Estimate all \( t \)-coordinates where \( P \) is maximum.
(c) What does a person hear in addition to the frequencies of 220, 330, and 440 Hz?
(d) Graph the pressure produced by a speaker that can vibrate at 110 Hz and above.

91. Daylight Hours in New Orleans
The seasonal variation in length of daylight can be modeled by a sine function. For example, the daily number of hours of daylight in New Orleans is given by
\[
h = \frac{35}{3} + \frac{7}{3} \sin \frac{2\pi x}{365},
\]
where \( x \) is the number of days after March 21 (disregarding leap year). (Source: Bushaw, Donald et al., *A Sourcebook of Applications of School Mathematics*. Copyright © 1980 by The Mathematical Association of America.)

(a) On what date will there be about 14 hr of daylight?
(b) What date has the least number of hours of daylight?
(c) When will there be about 10 hr of daylight?

(Modeling) Alternating Electric Current
The study of alternating electric current requires the solutions of equations of the form
\[
i = I_{\text{max}} \sin 2\pi ft,
\]
for time \( t \) in seconds, where \( i \) is instantaneous current in amperes, \( I_{\text{max}} \) is maximum current in amperes, and \( f \) is the number of cycles per second. (Source: Hannon, R. H., *Basic
90. (d)
For \( x = t \),
\[
P(t) = \sin[2\pi(110)t] + \frac{1}{2} \sin[2\pi(220)t] + \frac{1}{2} \sin[2\pi(330)t] + \frac{1}{2} \sin[2\pi(440)t]
\]

91. (a) 91.3 days after March 21, on June 20  (b) 273.8 days after March 21, on December 19  (c) 228.7 days after March 21, on November 4, and again after 318.8 days, on February 2  
92. .001 sec  
93. .0007 sec  
94. .004 sec  
95. .0014 sec  
96. 14°  
97. (a) \( \frac{1}{4} \) sec  
(b) \( \frac{1}{6} \) sec  
(c) .21 sec  
98. (a) 2 sec  
(b) 3 \( \frac{1}{3} \) sec  
99. (a) One such value is \( \frac{\pi}{3} \).  
(b) One such value is \( \frac{\pi}{4} \).  

### 7.6 Trigonometric Equations 669

**Technical Mathematics with Calculus, W. B. Saunders Company, 1978.** Find the smallest positive value of \( t \) given the following data.

92. \( i = 40, I_{\text{max}} = 100, f = 60 \)  
93. \( i = 50, I_{\text{max}} = 100, f = 120 \)  
94. \( i = I_{\text{max}}, f = 60 \)  
95. \( i = \frac{1}{2} I_{\text{max}}, f = 60 \)

(Modeling) Solve each problem.

96. **Accident Reconstruction**  The model
\[
.342D \cos \theta + h \cos^2 \theta = \frac{16D^2}{V_0^2}
\]

is used to reconstruct accidents in which a vehicle vaults into the air after hitting an obstruction. \( V_0 \) is velocity in feet per second of the vehicle when it hits, \( D \) is distance (in feet) from the obstruction to the landing point, and \( h \) is the difference in height (in feet) between landing point and takeoff point. Angle \( \theta \) is the takeoff angle, the angle between the horizontal and the path of the vehicle. Find \( \theta \) to the nearest degree if \( V_0 = 60, D = 80 \), and \( h = 2 \).

97. **Electromotive Force**  In an electric circuit, let
\[
V = \cos 2\pi t
\]

model the electromotive force in volts at \( t \) seconds. Find the smallest positive value of \( t \) where \( 0 \leq t \leq \frac{1}{2} \) for each value of \( V \).

(a) \( V = 0 \)  
(b) \( V = .5 \)  
(c) \( V = .25 \)

98. **Voltage Induced by a Coil of Wire**  A coil of wire rotating in a magnetic field induces a voltage modeled by
\[
e = 20 \sin \left( \frac{\pi t}{4} - \frac{\pi}{2} \right),
\]

where \( t \) is time in seconds. Find the smallest positive time to produce each voltage.

(a) 0  
(b) \( 10\sqrt{3} \)

99. **Movement of a Particle**  A particle moves along a straight line. The distance of the particle from the origin at time \( t \) is modeled by
\[
s(t) = \sin t + 2 \cos t.
\]

Find a value of \( t \) that satisfies each equation.

(a) \( s(t) = \frac{2 + \sqrt{3}}{2} \)  
(b) \( s(t) = \frac{3\sqrt{2}}{2} \)

100. Explain what is wrong with the following solution for all \( x \) over the interval \([0, 2\pi]\) of the equation \( \sin^2 x - \sin x = 0 \).

\[
\begin{align*}
\sin^2 x - \sin x &= 0 \\
\sin x - 1 &= 0 \quad \text{Divide by } \sin x. \\
\sin x &= 1 \\
&= \frac{\pi}{2} \\
x &= \frac{\pi}{2}
\end{align*}
\]

The solution set is \( \left\{ \frac{\pi}{2} \right\} \).
7.7 Equations Involving Inverse Trigonometric Functions

Until now, the equations in this chapter have involved trigonometric functions of angles or real numbers. Now we examine equations involving inverse trigonometric functions.

Solving for $x$ in Terms of $y$ Using Inverse Functions

**EXAMPLE 1** Solving an Equation for a Variable Using Inverse Notation

Solve $y = 3 \cos 2x$ for $x$.

**Solution** We want $\cos 2x$ alone on one side of the equation so we can solve for $2x$, and then for $x$.

\[
\frac{y}{3} = \cos 2x \quad \text{Divide by 3.}
\]

\[
2x = \arccos \frac{y}{3} \quad \text{Definition of arccosine (Section 7.5)}
\]

\[
x = \frac{1}{2} \arccos \frac{y}{3} \quad \text{Multiply by $\frac{1}{2}$.}
\]

Now try Exercise 7.

Solving Inverse Trigonometric Equations

**EXAMPLE 2** Solving an Equation Involving an Inverse Trigonometric Function

Solve $2 \arcsin x = \pi$.

**Solution** First solve for $\arcsin x$, and then for $x$.

\[
2 \arcsin x = \pi \quad \arcsin x = \frac{\pi}{2} \quad \text{Divide by 2.}
\]

\[
x = \sin \frac{\pi}{2} \quad \text{Definition of arcsine (Section 7.5)}
\]

\[
x = 1 \quad \text{(Section 6.2)}
\]

Verify that the solution satisfies the given equation. The solution set is $\{1\}$.

Now try Exercise 23.
EXAMPLE 3 Solving an Equation Involving Inverse Trigonometric Functions

Solve \( \cos^{-1} x = \sin^{-1} \frac{1}{2} \).

Solution \ Let \( \sin^{-1} \frac{1}{2} = u \). Then \( \sin u = \frac{1}{2} \) and for \( u \) in quadrant I, the equation becomes

\[
\cos^{-1} x = u
\]

\[
\cos u = x. \quad \text{Alternative form}
\]

Sketch a triangle and label it using the facts that \( u \) is in quadrant I and \( \sin u = \frac{1}{2} \). See Figure 31. Since \( x = \cos u \), \( x = \frac{\sqrt{3}}{2} \), and the solution set is \( \{\frac{\sqrt{3}}{2}\} \). Check.

Now try Exercise 29.

EXAMPLE 4 Solving an Inverse Trigonometric Equation Using an Identity

Solve \( \arcsin x - \arccos x = \frac{\pi}{6} \).

Solution \ Isolate one inverse function on one side of the equation.

\[
\arcsin x - \arccos x = \frac{\pi}{6}
\]

\[
\arcsin x = \arccos x + \frac{\pi}{6}
\]

Add \( \arccos x \). \ (1)

\[
\sin \left( \arccos x + \frac{\pi}{6} \right) = x
\]

Definition of \( \arcsin \)

Let \( u = \arccos x \), so \( 0 \leq u \leq \pi \) by definition.

\[
\sin \left( u + \frac{\pi}{6} \right) = x
\]

Substitute. \ (2)

\[
\sin \left( u + \frac{\pi}{6} \right) = \sin u \cos \frac{\pi}{6} + \cos u \sin \frac{\pi}{6}
\]

Sine sum identity (Section 7.3)

 Substitute this result into equation (2) to get

\[
\sin u \cos \frac{\pi}{6} + \cos u \sin \frac{\pi}{6} = x. \quad \text{(3)}
\]

From equation (1) and by the definition of the \( \arcsin \) function,

\[
-\frac{\pi}{2} \leq \arccos x + \frac{\pi}{6} \leq \frac{\pi}{2}
\]

\[
-2\pi \leq \arccos x \leq \frac{\pi}{3}.
\]

Subtract \( \pi \). (Section 1.7)

Since \( 0 \leq \arccos x \leq \pi \), we must have \( 0 \leq \arccos x \leq \frac{\pi}{3} \). Thus, \( x > 0 \), and we can sketch the triangle in Figure 32. From this triangle we find that \( \sin u = \sqrt{1-x^2} \).
Now substitute into equation (3) using \( \sin u = \sqrt{1 - x^2} \), \( \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \), and \( \cos u = x \).

\[
\sin u \cos \frac{\pi}{6} + \cos u \sin \frac{\pi}{6} = x
\]

\[
(\sqrt{1-x^2}) \frac{\sqrt{3}}{2} + x \cdot \frac{1}{2} = x
\]

Multiply by 2.

\[
(\sqrt{1-x^2})\sqrt{3} + x = 2x
\]

Subtract \( x \).

\[
(\sqrt{3})\sqrt{1-x^2} = x
\]

Square both sides. (Section 1.6)

\[
3(1-x^2) = x^2
\]

Distributive property (Section R.1)

\[
3 - 3x^2 = x^2
\]

\[
3 = 4x^2
\]

Add \( 3x^2 \).

\[
x = \sqrt{\frac{3}{4}}
\]

Solve for \( x \); Choose the positive square root because \( x > 0 \).

\[
x = \frac{\sqrt{3}}{2}
\]

Quotient rule (Section R.7)

To check, replace \( x \) with \( \frac{\sqrt{3}}{2} \) in the original equation:

\[
\arcsin \frac{\sqrt{3}}{2} - \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6},
\]

as required. The solution set is \( \{ \frac{\sqrt{3}}{2} \} \).

Now try Exercise 31.

### 7.7 Exercises

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<td>( x = \frac{1}{2} \arctan \frac{y}{3} )</td>
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**Concept Check**  
*Answer each question.*

1. Which one of the following equations has solution 0?
   - A. \( \arctan 1 = x \)  
   - B. \( \arccos 0 = x \)  
   - C. \( \arcsin 0 = x \)

2. Which one of the following equations has solution \( \frac{\pi}{4} \)?
   - A. \( \arcsin \frac{\sqrt{2}}{2} = x \)  
   - B. \( \arccos \left( -\frac{\sqrt{2}}{2} \right) = x \)  
   - C. \( \arctan \frac{\sqrt{3}}{3} = x \)

3. Which one of the following equations has solution \( \frac{3\pi}{4} \)?
   - A. \( \arctan 1 = x \)  
   - B. \( \arcsin \frac{\sqrt{2}}{2} = x \)  
   - C. \( \arccos \left( -\frac{\sqrt{2}}{2} \right) = x \)

4. Which one of the following equations has solution \( -\frac{\pi}{6} \)?
   - A. \( \arctan \frac{\sqrt{3}}{3} = x \)  
   - B. \( \arccos \left( -\frac{1}{2} \right) = x \)  
   - C. \( \arcsin \left( -\frac{1}{2} \right) = x \)
Solve each equation for x. See Example 1.

10. \( x = 2 \arcsin \frac{y}{3} \)
11. \( x = 4 \arcsin \frac{y}{6} \)
12. \( x = 3 \arcsin(-y) \)
13. \( x = \frac{1}{5} \arccos \left( \frac{-y}{2} \right) \)
14. \( x = \frac{1}{5} \arccot \frac{y}{3} \)
15. \( x = -3 + \arccos y \)
16. \( x = \frac{1}{2} (1 + \arctan y) \)
17. \( x = \arcsin(y + 2) \)
18. \( x = \arccot(y - 1) \)
19. \( x = \arcsin \left( \frac{y + 4}{2} \right) \)
20. \( x = \arccos \left( \frac{y - 4}{3} \right) \)
21. \( \{ -2\sqrt{2} \} \quad 24. \emptyset \)
22. \( (\pi - 3) \quad 26. \left\{ \frac{3\sqrt{2} + 2\pi}{6} \right\} \)
27. \( \left\{ \frac{3}{5} \right\} \quad 28. \left\{ \frac{12}{5} \right\} \quad 29. \left\{ \frac{4}{5} \right\} \)
30. \( \left\{ \frac{3}{4} \right\} \quad 31. \{0\} \quad 32. \left\{ \frac{\sqrt{3}}{2} \right\} \)
33. \( \left\{ \frac{1}{2} \right\} \quad 34. \emptyset \)
35. \( \left\{ -\frac{1}{2} \right\} \quad 36. \left\{ \frac{\sqrt{3}}{5} \right\} \)
37. \( \{0\} \quad 38. \{0\} \)
39. \( y = \arcsin X - \arccos X - \frac{\pi}{6} \)

40. \( Y_1 = \arcsin X - \arccos X \quad Y_2 = \frac{\pi}{6} \)

Solve each equation for exact solutions. See Examples 2 and 3.

23. \( \frac{4}{3} \cos^{-1} \frac{y}{4} = \pi \)
24. \( 4\pi + 4 \tan^{-1} y = \pi \)
25. \( 2 \arccos \left( \frac{y - \pi}{3} \right) = 2\pi \)
26. \( \arccos \left( y - \frac{\pi}{3} \right) = \frac{\pi}{6} \)
27. \( \arcsin x = \arctan \frac{3}{4} \)
28. \( \arctan x = \arccos \frac{5}{13} \)
29. \( \cos^{-1} x = \sin^{-1} \frac{3}{5} \)
30. \( \cot^{-1} x = \tan^{-1} \frac{4}{3} \)

Solve each equation for exact solutions. See Example 4.

31. \( \sin^{-1} x - \tan^{-1} 1 = -\frac{\pi}{4} \)
32. \( \sin^{-1} x + \tan^{-1} \sqrt{3} = \frac{2\pi}{3} \)
33. \( \arccos x + 2 \arcsin \frac{\sqrt{3}}{2} = \pi \)
34. \( \arccos x + 2 \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3} \)
35. \( \arcsin 2x + \arccos x = \frac{\pi}{6} \)
36. \( \arcsin 2x + \arccos x = \frac{\pi}{2} \)
37. \( \cos^{-1} x + \tan^{-1} x = \frac{\pi}{2} \)
38. \( \sin^{-1} x + \tan^{-1} x = 0 \)

39. Provide graphical support for the solution in Example 4 by showing that the graph of \( y = \arcsin x - \arccos x - \frac{\pi}{6} \) has x-intercept \( \sqrt{3} \approx .8660254 \). 
40. Provide graphical support for the solution in Example 4 by showing that the x-coordinate of the point of intersection of the graphs of \( Y_1 = \arcsin X - \arccos X \) and \( Y_2 = \frac{\pi}{6} \) is \( \frac{\sqrt{3}}{2} \approx .8660254 \).

The following equations cannot be solved by algebraic methods. Use a graphing calculator to find all solutions over the interval \([0, 6]\). Express solutions to as many decimal places as your calculator displays.

41. \( (\arctan x)^3 - x + 2 = 0 \)
42. \( \pi \sin^{-1}(2x) - 3 = -\sqrt{x} \)
43. (a) \[ A = .00506, \phi = .484; \]
\[ P = .00506 \sin(440\pi t + .484) \]
(b) The two graphs are the same.
\[
P(t) = .00506 \sin(440\pi t + .484)
\]
\[
P_1(t) + P_2(t) = .0012 \sin(440\pi t + .052) + .004 \sin(440\pi t + .61)
\]

44. (a) \[ A = .0035, \phi = .470; \]
\[ P = .0035 \sin(600\pi t + .47) \]
(b) The two graphs are the same.
\[
P(t) = .0035 \sin(600\pi t + .47)
\]
\[
P_1(t) + P_2(t) = .0025 \sin(600\pi t + \frac{\pi}{6}) + .001 \sin(600\pi t + \frac{\pi}{6})
\]

45. (a) \[ \tan \alpha = \frac{x}{z}; \tan \beta = \frac{x + y}{z} \]
(b) \[ \tan \alpha = \frac{x + y}{\tan \beta} \]
(c) \[ \alpha = \arctan \left( \frac{x \tan \beta}{x + y} \right) \]
(d) \[ \beta = \arctan \left( \frac{(x + y) \tan \alpha}{x} \right) \]

46. (a) \[ x = \sin u, -\frac{\pi}{2} \leq u \leq \frac{\pi}{2} \]
(b) \[ \tan u = \frac{x \sqrt{1 - x^2}}{1 - x^2} \]

43. **Tone Heard by a Listener**
When two sources located at different positions produce the same pure tone, the human ear will often hear one sound that is equal to the sum of the individual tones. Since the sources are at different locations, they will have different phase angles \( \phi \). If two speakers located at different positions produce pure tones \( P_1 = A_1 \sin(2\pi ft + \phi_1) \) and \( P_2 = A_2 \sin(2\pi ft + \phi_2) \), where \(-\frac{\pi}{2} \leq \phi_1, \phi_2 \leq \frac{\pi}{2}\), then the resulting tone heard by a listener can be written as \( P = A \sin(2\pi ft + \phi) \), where
\[
A = \sqrt{(A_1 \cos \phi_1 + A_2 \cos \phi_2)^2 + (A_1 \sin \phi_1 + A_2 \sin \phi_2)^2}
\]
and
\[
\phi = \arctan \left( \frac{A_1 \sin \phi_1 + A_2 \sin \phi_2}{A_1 \cos \phi_1 + A_2 \cos \phi_2} \right).
\]


(a) Calculate \( A \) and \( \phi \) if \( A_1 = .0012, \phi_1 = \frac{\pi}{7}, A_2 = .004, \phi_2 = .61 \). Also find an expression for \( P = A \sin(2\pi ft + \phi) \) if \( f = 220 \).
(b) Graph \( Y_1 = P \) and \( Y_2 = P_1 + P_2 \) on the same coordinate axes over the interval \([0, .01]\). Are the two graphs the same?

44. **Tone Heard by a Listener**
Repeat Exercise 43, with \( A_1 = .0025, \phi_1 = \frac{\pi}{7}, \)
\( A_2 = .001, \phi_2 = \frac{\pi}{2}, \) and \( f = 300 \).

45. **Depth of Field**
When a large-view camera is used to take a picture of an object that is not parallel to the film, the lens board should be tilted so that the planes containing the subject, the lens board, and the film intersect in a line. This gives the best “depth of field.” See the figure. (Source: Bushaw, Donald et al., *A Sourcebook of Applications of School Mathematics*. Copyright © 1980 by The Mathematical Association of America.)

(a) Write two equations, one relating \( \alpha, x, \) and \( z \), and the other relating \( \beta, x, y, \) and \( z \).
(b) Eliminate \( z \) from the equations in part (a) to get one equation relating \( \alpha, \beta, x, \) and \( y \).
(c) Solve the equation from part (b) for \( \alpha \).
(d) Solve the equation from part (b) for \( \beta \).

46. **Programming Language for Inverse Functions**
In Visual Basic, a widely used programming language for PCs, the only inverse trigonometric function available is arctangent. The other inverse trigonometric functions can be expressed in terms of arctangent as follows.

(a) Let \( u = \arcsin x \). Solve the equation for \( x \) in terms of \( u \).
(b) Use the result of part (a) to label the three sides of the triangle in the figure in terms of \( x \).
(c) Use the triangle from part (b) to write an equation for \( tan u \) in terms of \( x \).
(d) Solve the equation from part (c) for \( u \).
47. Alternating Electric Current  In the study of alternating electric current, instantaneous voltage is modeled by

\[ e = E_{\text{max}} \sin 2\pi ft, \]

where \( f \) is the number of cycles per second, \( E_{\text{max}} \) is the maximum voltage, and \( t \) is time in seconds.

(a) Solve the equation for \( t \).
(b) Find the smallest positive value of \( t \) if \( E_{\text{max}} = 12, e = 5 \), and \( f = 100 \). Use a calculator.

48. Viewing Angle of an Observer

While visiting a museum, Marsha Langlois views a painting that is 3 ft high and hangs 6 ft above the ground. See the figure. Assume her eyes are 5 ft above the ground, and let \( x \) be the distance from the spot where she is standing to the wall displaying the painting.

(a) Show that \( \theta \), the viewing angle subtended by the painting, is given by

\[ \theta = \tan^{-1} \left( \frac{4}{x} \right) - \tan^{-1} \left( \frac{1}{x} \right). \]

(b) Find the value of \( x \) for each value of \( \theta \).
(i) \( \theta = \frac{\pi}{6} \)  
(ii) \( \theta = \frac{\pi}{8} \)

(c) Find the value of \( \theta \) for each value of \( x \).
(i) \( x = 4 \)  
(ii) \( x = 3 \)

49. Movement of an Arm

In the exercises for Section 6.3 we found the equation

\[ y = \frac{1}{3} \sin \frac{4\pi t}{3}, \]

where \( t \) is time (in seconds) and \( y \) is the angle formed by a rhythmically moving arm.

(a) Solve the equation for \( t \).
(b) At what time(s) does the arm form an angle of .3 radian?

50. The function \( y = \sec^{-1} x \) is not found on graphing calculators. However, with some models it can be graphed as

\[ y = \frac{\pi}{2} - ((x > 0) - (x < 0)) \left( \frac{\pi}{2} - \tan^{-1}\left( \sqrt{x^2 - 1} \right) \right). \]

(This formula appears as \( Y_1 \) in the screen here.) Use the formula to obtain the graph of \( y = \sec^{-1} x \) in the window \([-4, 4] \) by \([0, \pi]\).
Chapter 7 Summary

NEW SYMBOLS

\[ \sin^{-1} x \ (\text{arcsin} \ x) \quad \text{inverse sine of } x \]
\[ \cos^{-1} x \ (\text{arccos} \ x) \quad \text{inverse cosine of } x \]
\[ \tan^{-1} x \ (\text{arctan} \ x) \quad \text{inverse tangent of } x \]

\[ \cot^{-1} x \ (\text{arccot} \ x) \quad \text{inverse cotangent of } x \]
\[ \sec^{-1} x \ (\text{arcsec} \ x) \quad \text{inverse secant of } x \]
\[ \csc^{-1} x \ (\text{arccsc} \ x) \quad \text{inverse cosecant of } x \]

QUICK REVIEW

CONCEPTS

7.1 Fundamental Identities

**Reciprocal Identities**

\[ \cot \theta = \frac{1}{\tan \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta} \]

**Quotient Identities**

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \]

**Pythagorean Identities**

\[ \sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \]

\[ 1 + \cot^2 \theta = \csc^2 \theta \]

**Negative-Angle Identities**

\[ \sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta \quad \tan(-\theta) = -\tan \theta \]
\[ \csc(-\theta) = -\csc \theta \quad \sec(-\theta) = \sec \theta \quad \cot(-\theta) = -\cot \theta \]

7.2 Verifying Trigonometric Identities

See the box titled Hints for Verifying Identities on page 613.

7.3 Sum and Difference Identities

**Cofunction Identities**

\[ \cos(90^\circ - \theta) = \sin \theta \quad \cot(90^\circ - \theta) = \tan \theta \]
\[ \sin(90^\circ - \theta) = \cos \theta \quad \sec(90^\circ - \theta) = \csc \theta \]
\[ \tan(90^\circ - \theta) = \cot \theta \quad \csc(90^\circ - \theta) = \sec \theta \]

**Sum and Difference Identities**

\[ \cos(A - B) = \cos A \cos B + \sin A \sin B \]
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \]
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B \]
\[ \sin(A - B) = \sin A \cos B - \cos A \sin B \]

Find a value of \( \theta \) such that \( \tan \theta = \cot 78^\circ \).

\[ \tan \theta = \cot 78^\circ \]
\[ \cot(90^\circ - \theta) = \cot 78^\circ \]
\[ 90^\circ - \theta = 78^\circ \]
\[ \theta = 12^\circ \]

Find the exact value of \( \cos(-15^\circ) \).

\[ \cos(-15^\circ) = \cos(30^\circ - 45^\circ) \]
\[ = \cos 30^\circ \cos 45^\circ + \sin 30^\circ \sin 45^\circ \]
\[ = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \]
\[ = \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4} \]
CONCEPTS

Sum and Difference Identities
\[
\begin{align*}
tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}
\end{align*}
\]

7.4 Double-Angle Identities and Half-Angle Identities

Double-Angle Identities
\[
\begin{align*}
cos 2A &= cos^2 A - sin^2 A \\
cos 2A &= 2 \cos^2 A - 1 \\
\sin 2A &= 2\sin A \cos A \\
n\tan 2A &= \frac{2 \tan A}{1 - \tan^2 A}
\end{align*}
\]

Product-to-Sum Identities
\[
\begin{align*}
\cos A \cos B &= \frac{1}{2}[\cos(A + B) + \cos(A - B)] \\
\sin A \sin B &= \frac{1}{2}[\cos(A - B) - \cos(A + B)] \\
\sin A \cos B &= \frac{1}{2}[\sin(A + B) + \sin(A - B)] \\
\cos A \sin B &= \frac{1}{2}[\sin(A + B) - \sin(A - B)]
\end{align*}
\]

Sum-to-Product Identities
\[
\begin{align*}
\sin A + \sin B &= 2 \sin\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\
\sin A - \sin B &= 2 \cos\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right) \\
\cos A + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\
\cos A - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right)
\end{align*}
\]

Half-Angle Identities
\[
\begin{align*}
\cos \frac{A}{2} &= \pm \sqrt{\frac{1 + \cos A}{2}} \\
\sin \frac{A}{2} &= \pm \sqrt{\frac{1 - \cos A}{2}} \\
\tan \frac{A}{2} &= \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \\
\tan \frac{A}{2} &= \frac{1 - \cos A}{\sin A} \quad \text{(The sign is chosen based on the quadrant of } \frac{A}{2})
\end{align*}
\]

EXAMPLES

Write \(\tan\left(\frac{\pi}{4} + \theta\right)\) in terms of \(\tan \theta\).
\[
\begin{align*}
\tan\left(\frac{\pi}{4} + \theta\right) &= \frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \\
&= \frac{1 + \tan \theta}{1 - \tan \theta} \\
\tan \frac{\pi}{4} &= 1
\end{align*}
\]

Given \(\cos \theta = -\frac{5}{13}\) and \(\sin \theta > 0\), find \(\sin 2\theta\).
Sketch a triangle in quadrant II and use it to find \(\sin \theta\):
\[
\sin \theta = \frac{12}{13}.
\]
\[
\begin{align*}
\sin 2\theta &= 2 \sin \theta \cos \theta \\
&= 2 \left(\frac{12}{13}\right) \left(-\frac{5}{13}\right) \\
&= -\frac{120}{169}
\end{align*}
\]

Write \((-\theta) \sin 2\theta\) as the difference of two functions.
\[
\begin{align*}
\sin(-\theta) \sin 2\theta &= \frac{1}{2} [\cos(-\theta - 2\theta) - \cos(-\theta + 2\theta)] \\
&= \frac{1}{2} [\cos(-3\theta) - \cos(\theta)] \\
&= \frac{1}{2} [\cos(3\theta) - \frac{1}{2} \cos \theta] \\
&= \frac{1}{2} \cos 3\theta - \frac{1}{2} \cos \theta
\end{align*}
\]

Write \(\cos \theta + \cos 3\theta\) as a product of two functions.
\[
\begin{align*}
\cos \theta + \cos 3\theta &= 2 \cos\left(\frac{\theta + 3\theta}{2}\right) \cos\left(\frac{\theta - 3\theta}{2}\right) \\
&= 2 \cos \left(\frac{4\theta}{2}\right) \cos\left(\frac{-2\theta}{2}\right) \\
&= 2 \cos 2\theta \cos(-\theta) \\
&= 2 \cos 2\theta \cos \theta
\end{align*}
\]

Find the exact value of \(\tan 67.5^\circ\).
We choose the last form with \(A = 135^\circ\).
\[
\tan 67.5^\circ = \tan \frac{135^\circ}{2} = \frac{1 - \cos 135^\circ}{\sin 135^\circ} = \frac{1 - \left(\frac{-\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}}
\]
\[
\begin{align*}
&= \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\
&= \frac{2 + \sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2} + 1
\end{align*}
\]

Rationalize the denominator; simplify.
CONCEPTS

7.5 Inverse Circular Functions

<table>
<thead>
<tr>
<th>Inverse Function</th>
<th>Domain</th>
<th>Interval</th>
<th>Quadrants of the Unit Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sin^{-1}x )</td>
<td>([-1, 1])</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
<td>I and IV</td>
</tr>
<tr>
<td>( y = \cos^{-1}x )</td>
<td>([-1, 1])</td>
<td>([0, \pi])</td>
<td>I and II</td>
</tr>
<tr>
<td>( y = \tan^{-1}x )</td>
<td>((-\infty, \infty))</td>
<td>((-\frac{\pi}{2}, \frac{\pi}{2}))</td>
<td>I and IV</td>
</tr>
<tr>
<td>( y = \cot^{-1}x )</td>
<td>((-\infty, \infty))</td>
<td>((0, \pi))</td>
<td>I and II</td>
</tr>
<tr>
<td>( y = \sec^{-1}x )</td>
<td>((-\infty, -1) \cup [1, \infty))</td>
<td>([0, \pi], y \neq \frac{\pi}{2})</td>
<td>I and II</td>
</tr>
<tr>
<td>( y = \csc^{-1}x )</td>
<td>((-\infty, -1) \cup [1, \infty))</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}), y \neq 0)</td>
<td>I and IV</td>
</tr>
</tbody>
</table>

Evaluate \( y = \cos^{-1}0 \).
Write \( y = \cos^{-1}0 \) as \( \cos y = 0 \). Then \( y = \frac{\pi}{2} \), because \( \cos \frac{\pi}{2} = 0 \) and \( \frac{\pi}{2} \) is in the range of \( \cos^{-1}x \).

Evaluate \( \sin[\tan^{-1}\left(-\frac{3}{4}\right)] \).
Let \( u = \tan^{-1}\left(-\frac{3}{4}\right) \). Then \( \tan u = -\frac{3}{4} \). Since \( \tan^{-1}x \) is negative in quadrant IV, sketch a triangle as shown.

We want \( \sin[\tan^{-1}\left(-\frac{3}{4}\right)] = \sin u \). From the triangle, \( \sin u = -\frac{3}{5} \).

7.6 Trigonometric Equations

Solving a Trigonometric Equation

1. Decide whether the equation is linear or quadratic in form, so you can determine the solution method.
2. If only one trigonometric function is present, first solve the equation for that function.
3. If more than one trigonometric function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to 0 to solve.
4. If the equation is quadratic in form, but not factorable, use the quadratic formula. Check that solutions are in the desired interval.
5. Try using identities to change the form of the equation. It may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions.

Solve \( \tan \theta + \sqrt{3} = 2\sqrt{3} \) over the interval \([0^\circ, 360^\circ]\).
Use a linear method.

\[
\tan \theta + \sqrt{3} = 2\sqrt{3} \\
\tan \theta = \sqrt{3} \\
\theta = 60^\circ
\]

Another solution over \([0^\circ, 360^\circ]\) is

\[
\theta = 60^\circ + 180^\circ = 240^\circ
\]

The solution set is \(\{60^\circ, 240^\circ\}\).
## Chapter 7 Review Exercises

6. E  7. I  8. \( \frac{\cos^2 \theta}{\sin \theta} \)  
9. \( \frac{1}{\cos^2 \theta} \)  10. \( \frac{1 + \cos \theta}{\sin \theta} \)  
11. \( \sin x = -\frac{4}{5}; \tan x = -\frac{4}{3}; \cot(-x) = \frac{3}{4} \)  
12. \( \cot x = -\frac{4}{3}; \csc x = \frac{\sqrt{41}}{5}; \sec x = -\frac{\sqrt{41}}{4} \)  
24. \( \frac{4 - 9\sqrt{51}}{12\sqrt{11} - 3} \)  25. \( \frac{1}{2} \)  
26. \( \frac{\sqrt{6}}{3} \)  

### Concept Check

For each expression in Column I, choose the expression from Column II that completes an identity.

<table>
<thead>
<tr>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \sec x = )</td>
<td>A. ( \frac{1}{\sin x} )</td>
</tr>
<tr>
<td>2. ( \csc x = )</td>
<td>B. ( \frac{1}{\cos x} )</td>
</tr>
<tr>
<td>3. ( \tan x = )</td>
<td>C. ( \frac{\sin x}{\cos x} )</td>
</tr>
<tr>
<td>4. ( \cot x = )</td>
<td>D. ( \frac{1}{\cot^2 x} )</td>
</tr>
<tr>
<td>5. ( \tan^2 x = )</td>
<td>E. ( \frac{1}{\cos^2 x} )</td>
</tr>
<tr>
<td>6. ( \sec^2 x = )</td>
<td>F. ( \frac{\cos x}{\sin x} )</td>
</tr>
</tbody>
</table>

### Use identities to write each expression in terms of \( \sin \theta \) and \( \cos \theta \), and simplify.

7. \( \sec^2 \theta - \tan^2 \theta \)  
8. \( \frac{\cot \theta}{\sec \theta} \)  
9. \( \tan^2 \theta(1 + \cot^2 \theta) \)  
10. \( \csc \theta + \cot \theta \)  
11. Use the trigonometric identities to find \( \sin x \), \( \tan x \), and \( \cot(-x) \), given \( \cos x = \frac{3}{5} \) and \( x \) is in quadrant IV.  
12. Given \( \tan x = -\frac{3}{7} \), where \( \frac{\pi}{2} < x < \pi \), use the trigonometric identities to find \( \cot x \), \( \csc x \), and \( \sec x \).

### Concept Check

For each expression in Column I, use an identity to choose an expression from Column II with the same value.

<table>
<thead>
<tr>
<th>Column I</th>
<th>Column II</th>
</tr>
</thead>
<tbody>
<tr>
<td>13. ( \cos 210^\circ )</td>
<td>A. ( \sin(-35^\circ) )</td>
</tr>
<tr>
<td>14. ( \sin 35^\circ )</td>
<td>B. ( \cos 55^\circ )</td>
</tr>
<tr>
<td>15. ( \tan(-35^\circ) )</td>
<td>C. ( \frac{1 + \cos 150^\circ}{2} )</td>
</tr>
<tr>
<td>16. ( -\sin 35^\circ )</td>
<td>D. ( 2 \sin 150^\circ \cos 150^\circ )</td>
</tr>
<tr>
<td>17. ( \cos 35^\circ )</td>
<td>E. ( \cos 150^\circ \cos 60^\circ - \sin 150^\circ \sin 60^\circ )</td>
</tr>
<tr>
<td>18. ( \cos 75^\circ )</td>
<td>F. ( \cot(-35^\circ) )</td>
</tr>
<tr>
<td>19. ( \sin 75^\circ )</td>
<td>G. ( \cos^2 150^\circ - \sin^2 150^\circ )</td>
</tr>
<tr>
<td>20. ( \sin 300^\circ )</td>
<td>H. ( \sin 15^\circ \cos 60^\circ + \cos 15^\circ \sin 60^\circ )</td>
</tr>
<tr>
<td>21. ( \cos 300^\circ )</td>
<td>I. ( \cos(-35^\circ) )</td>
</tr>
<tr>
<td>22. ( \cos(-55^\circ) )</td>
<td>J. ( \cot 125^\circ )</td>
</tr>
</tbody>
</table>

For each of the following, find \( \sin(x + y) \), \( \cos(x - y) \), \( \tan(x + y) \), and the quadrant of \( x + y \).

23. \( \sin x = -\frac{1}{4} \), \( \cos y = -\frac{4}{5} \), \( x \) and \( y \) in quadrant III  
24. \( \sin x = \frac{1}{10} \), \( \cos y = \frac{4}{5} \), \( x \) in quadrant I, \( y \) in quadrant IV  

Find each of the following.

25. \( \cos \frac{\theta}{2} \), given \( \cos \theta = -\frac{1}{2} \), with \( 90^\circ < \theta < 180^\circ \)  
26. \( \sin y \), given \( \cos 2y = -\frac{1}{3} \), with \( \frac{\pi}{2} < y < \pi \)
Graph each expression and use the graph to conjecture an identity. Then verify your conjecture algebraically.

27. \( \frac{\sin 2x + \sin x}{\cos 2x - \cos x} = \cot \frac{x}{2} \)
28. \( \frac{1 - \cos 2x}{\sin 2x} = \tan x \)
30. \( \frac{2 \cos^2 x - \cos x}{\cos^2 x - \sin^2 x} = \frac{\cos^2 x - \sin^2 x}{\sec x} \)

Verify that each equation is an identity.

29. \( \sin^2 x - \sin^2 y = \cos y - \cos^2 x \)
31. \( \frac{\sin^2 x}{2 - 2 \cos x} = \cos^2 \frac{x}{2} \)
32. \( \frac{\sin 2x}{\sin x} = \frac{2}{\sec x} \)
33. \( 2 \cos A - \sec A = \cos A - \tan A \cdot \csc A \)
34. \( \frac{2 \cos B}{\sec 2B} = \sec B \)
35. \( 1 + \tan^2 \alpha = 2 \tan \alpha \cdot \csc 2 \alpha \)
36. \( \frac{2 \cot x}{\tan 2x} = \csc^2 x - 2 \)
37. \( \tan \theta \sin 2\theta = 2 - 2 \cos^2 \theta \)
38. \( \csc A \sin 2A - \sec A = \cos 2A \cdot \sec A \)
39. \( 2 \tan x \csc 2x - \tan^2 x = 1 \)
40. \( 2 \cos^2 \theta - 1 = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \)
41. \( \tan \theta \cos^2 \theta = \frac{2 \tan \theta \cos^2 \theta - \tan \theta}{1 - \tan^2 \theta} \)
42. \( \sec^2 \alpha - 1 = \frac{\sec 2\alpha - 1}{\sec 2\alpha + 1} \)
43. \( 2 \cos^3 x - \cos x = \frac{\cos^2 x - \sin^2 x}{\sec x} \)
44. \( \sin^3 \theta = \sin \theta - \cos^2 \theta \sin \theta \)

Give the exact real number value of \( y \). Do not use a calculator.

45. \( y = \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) \)
46. \( y = \arccos \left( \frac{1}{2} \right) \)
47. \( y = \tan^{-1} (-\sqrt{3}) \)
48. \( y = \arcsin (-1) \)
49. \( y = \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \)
50. \( y = \arctan \left( \frac{\sqrt{3}}{3} \right) \)
51. \( y = \sec^{-1} (-2) \)
52. \( y = \arccsc \left( \frac{2\sqrt{3}}{3} \right) \)
53. \( y = \arccot (-1) \)

Give the degree measure of \( \theta \). Do not use a calculator.

54. \( \theta = \arccos \left( \frac{1}{2} \right) \)
55. \( \theta = \arcsin \left( \frac{\sqrt{3}}{2} \right) \)
56. \( \theta = \tan^{-1} 0 \)

Use a calculator to give the degree measure of \( \theta \).

57. \( \theta = \arctan 1.7804675 \)
58. \( \theta = \sin^{-1} (-0.66045320) \)
59. \( \theta = \cos^{-1} 0.80396577 \)
60. \( \theta = \cot^{-1} 4.5046388 \)
61. \( \theta = \arccsc 3.4723155 \)
62. \( \theta = \csc^{-1} 7.4890096 \)

Evaluate the following without using a calculator.

63. \( \cos(\arccos (-1)) \)
64. \( \sin \left( \arcsin \left( \frac{-\sqrt{3}}{2} \right) \right) \)
65. \( \arccos \left( \cos \frac{3\pi}{4} \right) \)
66. \( \arccsc(\sec \pi) \)
67. \( \tan^{-1} \left( \tan \frac{\pi}{4} \right) \)
68. \( \cos^{-1}(\cos 0) \)
69. \( \sin \left( \arccos \frac{3}{4} \right) \)
70. \( \cos(\arctan 3) \)
71. \( \cos(\csc^{-1} (-2)) \)
8.6602567 ft; There may

98.

97.

96.

95.

94.

93.

92.

91.
80. June 20

79. 

78. 

77.

76.

75.

74.

73.

72.

Write each of the following as an algebraic (nontrigonometric) expression in u.

Solve each equation for solutions over the interval \([0, 2\pi]\).

Give all solutions for each equation.

Solve each equation for solutions over the interval \([0^\circ, 360^\circ]\). If necessary, express solutions to the nearest tenth of a degree.

Solve each equation in Exercises 91–97 for \(x\).

(Modeling) Solve each problem.

99. Viewing Angle of an Observer A 10-ft-wide chalkboard is situated 5 ft from the left wall of a classroom. See the figure. A student sitting next to the wall \(x\) feet from the front of the classroom has a viewing angle of \(\theta\) radians.

(a) Show that the value of \(\theta\) is given by the function defined by

\[
f(x) = \arctan\left(\frac{15}{x}\right) - \arctan\left(\frac{5}{x}\right).
\]

(b) Graph \(f(x)\) with a graphing calculator to estimate the value of \(x\) that maximizes the viewing angle.
100. Snell’s Law Recall Snell’s law from Exercises 102 and 103 of Section 5.3:

\[ \frac{c_1}{c_2} = \frac{\sin \theta_1}{\sin \theta_2}, \]

where \( c_1 \) is the speed of light in one medium, \( c_2 \) is the speed of light in a second medium, and \( \theta_1 \) and \( \theta_2 \) are the angles shown in the figure. Suppose a light is shining up through water into the air as in the figure. As \( \theta_1 \) increases, \( \theta_2 \) approaches 90°, at which point no light will emerge from the water. Assume the ratio \( \frac{c_1}{c_2} \) in this case is .752. For what value of \( \theta_1 \) does \( \theta_2 = 90° \)? This value of \( \theta_1 \) is called the critical angle for water.

101. Snell’s Law Refer to Exercise 100. What happens when \( \theta_1 \) is greater than the critical angle?

102. British Nautical Mile The British nautical mile is defined as the length of a minute of arc of a meridian. Since Earth is flat at its poles, the nautical mile, in feet, is given by

\[ L = 6077 - 31 \cos \theta, \]

where \( \theta \) is the latitude in degrees. See the figure. (Source: Bushaw, Donald et al., A Sourcebook of Applications of School Mathematics. Copyright © 1980 by The Mathematical Association of America.)

(a) Find the latitude between 0° and 90° at which the nautical mile is 6074 ft.
(b) At what latitude between 0° and 180° is the nautical mile 6108 ft?
(c) In the United States, the nautical mile is defined everywhere as 6080.2 ft. At what latitude between 0° and 90° does this agree with the British nautical mile?

103. The function \( y = \csc^{-1} x \) is not found on graphing calculators. However, with some models it can be graphed as

\[ y = (x > 0) - (x < 0) \left( \frac{\pi}{2} - \tan^{-1} \left( \sqrt{x^2 - 1} \right) \right). \]

(This formula appears as \( Y_1 \) in the screen here.) Use the formula to obtain the graph of \( y = \csc^{-1} x \) in the window \([-4, 4] \) by \([-\frac{\pi}{2}, \frac{\pi}{2}] \).

104. (a) Use the graph of \( y = \sin^{-1} x \) to approximate \( \sin^{-1} .4 \).
(b) Use the inverse sine key of a graphing calculator to approximate \( \sin^{-1} .4 \).

---

### Chapter 7 Test

1. Given \( \tan x = -\frac{3}{5}, \frac{3\pi}{2} < x < 2\pi \), use trigonometric identities to find \( \sin x \) and \( \cos x \).
2. Express \( \tan^2 x - \sec^2 x \) in terms of \( \sin x \) and \( \cos x \), and simplify.
3. Find \( \sin(x + y) \), \( \cos(x - y) \), and \( \tan(x + y) \), if \( \sin x = -\frac{1}{3}, \cos y = -\frac{2}{3} \), \( x \) is in quadrant III, and \( y \) is in quadrant II.
4. Use a half-angle identity to find \( \sin(-22.5°) \).
3. \( \sin(x + y) = \frac{2 - 2\sqrt{42}}{15} \),
\( \cos(x - y) = \frac{4\sqrt{3} - \sqrt{27}}{15} \),
\( \tan(x + y) = \frac{2\sqrt{3} - 4\sqrt{21}}{8 + \sqrt{42}} \).

4. \( \frac{-\sqrt{2} - \sqrt{2}}{2} \)

5. \( \sec x - \sin x \tan x = \cos x \)

6. \( \cot \left( \frac{x}{2} \right) - \cot x = \csc x \)

9. (a) \( \sin \theta \)  (b) \( -\sin \theta \)

10. (a) \( V = 163 \cos \left( \frac{\pi}{2} - \omega t \right) \)
(b) 163 volts; \( \frac{1}{240} \) sec

11. \([-1, 1]: \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \)

12. (a) \( \frac{2\pi}{3} \)  (b) \( -\frac{\pi}{3} \)

(c) 0  (d) \( \frac{2\pi}{3} \)

13. (a) \( \frac{\sqrt{3}}{3} \)  (b) \( \frac{4\sqrt{2}}{9} \)

15. \( \{90^\circ, 270^\circ\} \)

16. \( \{18.4^\circ, 135^\circ, 198.4^\circ, 315^\circ\} \)

18. \( \frac{2\pi}{3} + 4n\pi, \frac{4\pi}{3} + 4n\pi, \) where \( n \) is any integer

19. (a) \( x = \frac{1}{3} \arccos y \)
(b) \( \left\{ \frac{4}{5} \right\} \)

20. (Modeling) Movement of a Runner’s Arm A runner’s arm swings rhythmically according to the model
\[ y = \frac{\pi}{8} \cos \left[ \pi \left( t - \frac{1}{3} \right) \right] , \]
where \( y \) represents the angle between the actual position of the upper arm and the downward vertical position and \( t \) represents time in seconds. At what times over the interval \([0, 3]\) is \( y \) equal to 0?
How can we determine the amount of oil in a submerged storage tank?
The level of oil in a storage tank buried in the ground can be found in much the same way a dipstick is used to determine the oil level in an automobile crankcase. The person in the figure on the left has lowered a calibrated rod into an oil storage tank. When the rod is removed, the reading on the rod can be used with the dimensions of the storage tank to calculate the amount of oil in the tank.

Suppose the ends of the cylindrical storage tank in the figure are circles of radius 3 ft and the cylinder is 20 ft long. Determine the volume of oil in the tank if the rod shows a depth of 2 ft. *(Hint: The volume will be 20 times the area of the shaded segment of the circle shown in the figure on the right.)*

\[
20 \left[ 9 \arctan\left(\frac{\sqrt{8}}{3}\right) - \sqrt{8} \right] \approx 165 \text{ ft}^3
\]