Instructor's Resource Manual

for use with

Chapter Zero—
Fundamental Notions of Abstract Mathematics, 2e

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About This Manual

Dear Colleague:

My intention is that this be a very down-to-earth, practical guide to help you when you use *Chapter Zero* in the classroom. *Chapter Zero* is written in such a way that its ultimate worth as a learning tool depends a great deal on the way that students interact with it, with each other, and with you. The dynamics of each individual course strongly influences how successful the text will be. Therefore, I write this guide almost entirely from the point of view of a teacher who has used *Chapter Zero* in a class and only occasionally from the point of view of the author.

I have used the first edition of *Chapter Zero* in my own classes for a number of years. I also received detailed feedback from reviewers that had used the first edition in their classes. With this information in hand, I have made a number of (what I think are) improvements in the second edition. I hope that these changes will make things run more smoothly for instructors—including me—that use the book.

The Instructor's Resource Guide is divided into two major parts. In the first part, I will speak in detail about the way that I organize and run the class when I use *Chapter Zero*. I will also discuss specific difficulties that I have encountered and suggest strategies for heading them off or dealing with them if they arise. (There are a few additions from the first edition of the IRM, but it is largely the same as the corresponding section in the first edition.)

The discussion in the first part will concentrate on the particular model that I have used for running the class: a seminar-like model in which there is virtually no lecturing. Class time is spent in discussion, small group work, and with students presenting their proofs and solutions to one another. The instructor acts primarily as a moderator. It makes sense to think of this as the "native" model for using the book; *Chapter Zero* was written precisely for use in this way. However, it is not at all difficult to see ways of including more lecturing, and you may choose to do so. Even if you do things in a way very different from mine, some of my comments may be useful to you; so I urge you at least to skim the first part.

In the second part, I will go through the book chapter by chapter and make specific comments. I will do as many of the following as seem useful:

- Tell you what I do when I cover the chapter.
- Point out specific places where my students have encountered difficulties (and ways that I have found of helping them through).
- Talk about overall strategies I have used to make the chapter go more smoothly.

- Make suggestions for class discussion.

At the end of the second part, I include a section elaborating on the dependency chart that appears in the front matter to *Chapter Zero*, as well as a list of the errors that I have found in the book. As I become aware of other errors, I will update this section in the web-version of the IRM. Known errors will be corrected for the second and subsequent printings. If you find any errors, I would consider it a great kindness were you to let me know about them.

Please understand that what you find herein are thoughts based on my own experiences teaching out of *Chapter Zero*. I am continually learning new things as I teach, and therefore I have no pretensions that what I write is the definitive or final word. I hope that many of my comments will be useful to you. There is no doubt that some of them will not be. If, in using the book, you encounter difficulties that I do not address in this guide, I hope that you will feel free to contact me. I will try to help if I can. Better yet, if you find additional strategies that work in your classes or for your students, I would be grateful if you would let me know of them. That way I can use them myself and pass them on to other colleagues who are using the book.

Respectfully,

Carol S. Schumacher

Part I

A Course that uses *Chapter Zero*

This chapter describes how I have used *Chapter Zero* in a sophomore-level course at Kenyon College. I include fairly detailed descriptions of the day-to-day routine of the class, various problems and pitfalls I have encountered, and my general strategies for dealing with them. The course is typically taken by students who have had three semesters of calculus (though no knowledge of calculus is assumed by the text), and is a prerequisite for our upper-level courses in abstract algebra, real analysis, etc. Each year a number of our best prepared and most mathematically inclined first-year students take the course. They tend to do very well, often outperforming the sophomores. Recently, almost half of the students have been first-year students. For the purposes of this discussion, I will call the course "Foundations."

Remark: The *Note to the Student* found in the front matter of *Chapter Zero* contains essential information about the way that students are meant to use the text and about the meanings of various labels that are used through out the book. It is a "must read" for teachers as well as students who use the book.

Philosophy

Goals

At root, my goal in Foundations is quite simple. Students who complete the course should be well-prepared to succeed in more advanced studies of abstract mathematics. This means that they must have acquired certain skills:

- They should know how to make sense of an abstract definition by analyzing it carefully and constructing examples.
- They should also know how to make sense of a mathematical statement, and be able to bring to bear a variety of strategies for constructing its proof.
- They must be able to recognize a rigorous proof when they read one. Conversely, they need to be able to pick out the weak spot(s) in a less rigorous argument—their own or someone else's.
- They ought to be able to fill in details in a sketchy proof.
- Once they have devised a proof, they must be able to write it down in a clear, concise manner using correct English and mathematical grammar.
- Students have to be able to present and defend a proof to a group of their peers—oral communication is as important as written communication.

Moreover, I hope that they will think of these as central tools in the mathematical enterprise. (In reality, of course, these are very sophisticated skills, so students who finish *Foundations* are only beginning to fully acquire these skills; the *Foundations* course is a first step, but a very important one.)

However, students need more than an array of desirable skills. They also need some core knowledge to work from. Thus, one of my aims in the Foundations course is for my students to acquire a thorough-going mastery of some fundamental topics. The topics in *Chapter Zero* underlie and permeate virtually all branches of mathematics, and students will see them over and over again in their further studies.

Learning a Language

I often think of Foundations as a language course. Learning mathematical language goes beyond just learning jargon (though there is plenty of vocabulary to acquire). Mathematical language is far more precise than ordinary ways of speaking and writing, and using language so precisely does not come naturally to most people. Yet learning this language (and learning how to use it to accomplish a given task) is an essential step on the road to mathematical maturity.

Speaking a new language differs greatly from reading it, writing it or listening to someone else speak it. Getting students to talk seriously about mathematics is therefore an essential part of Foundations. I conduct the course seminar-style. I give very few (if any) lectures, acting primarily as a moderator. The students do most of the talking, presenting proofs and exercises to one another or working in small groups. As I say in the preface to *Chapter Zero*, "*the most important lines of communication are between students.*" In general, I find that the less I talk in a class, the more the students get out of it.

Student Autonomy

One of my principal ambitions as a teacher is to finally make my students independent of me. Nothing else that I teach them will be half so valuable or powerful as the ability to reach conclusions by reasoning logically from first principles and being able to justify those conclusions in clear, persuasive language (either oral or written).

Furthermore, I want my students to experience the unmistakable feeling that comes when one really understands something thoroughly.[1] Much "classroom knowledge" is fairly superficial, and students often find it hard to judge their own level of understanding. For many students, the only way they know whether they are "getting it" comes from the grade they make on an exam. I hope that my own students, by passing beyond superficial acquaintance with some mathematical ideas, will become less reliant on such externals. When they can distinguish between really *knowing* something and merely knowing *about* something, they will be on their way to becoming independent learners.

There are also some basic skills that set students free to learn on their own. I believe they must be able to read and understand books that contain new knowledge. They must be able to handle problems that aren't "pre-digested" by learning to ask their own questions and having some tools with which to find

[1] Every year I have Foundations students come to tell me that for the first time they have "really nailed" a proof. They have no doubts that it is right and are ecstatic! They are experiencing *the feeling* for the first time. It is clear that they will never forget it.

answers to those questions. They must be able to distinguish the essential from the trivial.[2]

So what are the means by which students gain this sort of autonomy? I find that students become more autonomous when they are given real opportunities to do things on their own. I wrote *Chapter Zero* with precisely this in mind. Both as an author and as a teacher, I do my best to help students find their own answers rather than answering their questions for them. However, no matter how hard I try to make this happen, it will not happen unless I allow the Foundations class move at the students' pace, instead of my own. This demands that I do two related things that I find difficult. First, I have to fight my tendency to become impatient; and second, I have to learn when to keep my counsel and let the students do the talking. It is easy to become impatient, since in a couple of lectures I could "cover" material that can occupy my students for four or five class meetings. Letting students work through things on their own always takes longer, but they come away being able to devise their own arguments and with a better grasp of the ideas. That, after all, is the whole point!

It is also easy to talk too much during class. I think Foundations students need to learn to talk to each other more than they talk to me. When conversing with me, students inevitably look to me for answers. In a conversation with their peers, students know they have to find their own answers. Fostering this sort of conversation also requires patience on my part. Frequently my students will wrestle with an issue that I could rapidly clarify with a few words or a well-chosen example. However, the most valuable class meetings are usually those in which I manage to rein in my impulse to jump to their rescue. The students begin to talk to each other as they try to resolve the issue. Sometimes they succeed and sometimes they do not, but they learn much from each other and from their own struggles to state their points of view convincingly. If in the end they reach a deadlock, or if something more needs to be said, then I can still add my own remarks to the discussion.

I grant my Foundations students as much autonomy as I think they can handle. This does not mean that I let them take charge of the class. It means that what happens in the class happens because they make it happen.

[2]Naturally, this is not the work of any single course. It is a process. However, I try to make Foundations the first serious step in this direction.

Class Mechanics

Class Organization

Since Foundations is an unusual kind of class, I think it is worthwhile to tell you in some detail how it is organized—what I do and exactly what I expect from my students (both in and out of class).

Student Responsibilities

Because of my goal of student autonomy, much of the responsibility for moving the class along rests with the students.

Preparing for Class Since I seldom lecture, it is the students' regular responsibility to read the textbook carefully. I expect them to work out all of the examples and exercises as they read through the section.[3] These are a big part of coming to understand the reading.

I assign explicit problems and theorems for the students to work on outside of class. Though I know that not all students will "get" all of these, I expect that each student will regularly be proving theorems and will have worked on every assigned problem enough to understand its statement, to have in mind the relevant definitions, and to comprehend the mathematical issues at hand.

Class routine We go over exercises and examples during class discussions, as needed. In the process, we can conveniently discuss and clarify new definitions. Though I may have a specific goal in mind, I try to *moderate* rather than *lead* the discussion. The direction that the class discussion takes chiefly depends on the students. During class discussions, I call on students by name to give their results for exercises and examples.

Students are responsible for presenting problems and theorems at the board. This is usually done by volunteers, who receive credit for their presentations.

During a presentation, the rest of the class is not off the hook just because another student volunteered! Those who are sitting down are responsible for contributing questions and comments that help clarify what is being presented at the board. I try to make it clear that questions and comments

[3]These are never assigned explicitly. Exercises and examples are understood to be part of everyone's assignment, unless otherwise noted.

need not only come from students who don't understand the problem. Part of the purpose of class participation, I tell them, is to help the person who is presenting work explain things as clearly as possible. Another student who has done the problem and understands it well can often make suggestions that help clarify the wording or the structure of a presentation. When students are not asking questions that I think they ought to be asking, I frequently call on students that are sitting down and ask them the questions pointed questions about the presentation.

Written Work I give regular written assignments, though they are not as frequent as they would be in a lecture course. (Since students are regularly participating in the Foundations class, I get regular feedback about their work and progress.) Written assignments come in two major varieties:

- Theorems or problems that I want to make sure every student in the class works through. This may be because an important mathematical idea is involved, or because the solution requires a particular proof technique that I want to be sure every student is mastering. Sometimes I assign a problem or theorm just because it is obvious no one in the class has worked on it. For instance, I always give an assignment on mathematical induction in which the students must apply it in more than one context. I permit students to work cooperatively on these problems. (See "Cooperative Learning" below.)
- Take-home exams in which students must work individually and prove theorems that they have not seen before. For these, the students are allowed to talk to me, but not to anyone else. They are allowed to use the text and any notes that they have taken for the class. They are not allowed to consult outside sources.

Naturally, I require that written work be carefully prepared and turned in on time.

Class Presentations

Though the atmosphere in the Foundations class is informal and friendly, what we do in the class is serious business. In particular, the presentations made by students are taken very seriously since they spearhead the work of the class.

Here are some of the things my students need to know about making a presentation at the board:

- In order to make the presentation go smoothly, the presenter needs to have written out the proof in detail and gone over the major ideas and transitions, so that he or she can make clear the path of the proof to others.

- The purpose of class presentation is not to prove to the instructor that the presenter has done the problem. It is to make the ideas of the proof clear to the other students.

- Presenters are to write in complete sentences, using proper English and mathematical grammar.

- Presenters should explain their reasoning as they go along, not simply write everything down and then turn to explain.

- Fellow students are allowed to ask questions at any point and it is the responsibility of the person making the presentation to answer those questions to the best of his or her ability.

- Since the presentation is directed at the students, the presenter should frequently make eye-contact with the students in order to address questions when they arise and also be able to see how well the other students are following the presentation.

I grade the students at the board both on the content of their proof and on the quality of the presentation. I grade the quality more harshly as the semester wears on, since I expect the students' presentation skills to develop with practice.

Remark: Any time a problem lends itself to being discussed orally, I try to deal with it in this way. I have two reasons for this. First of all, oral discussion usually takes less time than class presentation. I always work to gain time wherever I can without rushing the subtler ideas. The second and more important factor I learned the hard way. If I have students present problems in which the reasoning is simple enough to convey without writing anything down, students get lazy about writing in complete sentences and giving a careful presentation. (It is hard to fault students here, since belaboring the obvious does seem an unnecessary waste of time.) I tried the tactic of letting my students be a bit more informal in such presentations, but found in the end that this sloppiness spilled over into trickier problems, where it *was not* acceptable. Now I think carefully through the exercises, examples, and problems, dividing them in my mind into oral problems and written problems. Things go more smoothly when I do this.

Cooperative Learning

I encourage my students to work together outside of class. The kind of material that they encounter lends itself very well to give and take, and students benefit from being able to bounce ideas off of each other. I think that most students who thrive in the course are part of a small group of 3-4 students who work together regularly outside of class. As an added benefit, I think that students who work in a small group typically enjoy the class more. The intense working sessions cement friendships that go beyond the work in Foundations.

Students are not, of course, allowed to work together on exams. They are also not permitted to write up written assignments together. In the handout that I prepare for the first day of class, I say explicitly that "all written work must finally be [the student's] own expression." This prevents a weaker student from relying too much on a friend who is a stronger student. Students know that after talking things out with their friends they will have to write solutions up on their own; therefore, they must thoroughly understand them. (Some students ignore this instruction at first, but if I see papers that look too similar, I remind them of it. This usually solves the problem.)

My Role

I hesitate—even more than usual—as I write this section. I find it hard to describe precisely what I do, for I spend much of my time *responding* to the changing situation in the class. I am often guided by instinct. This is probably true of anyone who teaches a class in which the students move things along.

On the other hand, it is certainly possible to make some general remarks about what my role in the Foundations class. I must make the obvious kinds of decisions:

- What topics should I cover?
- What specific problems and theorems shall I assign?
- Which problems (word used broadly!) should be covered orally, which should be presented at the board, which should be part of a written assignment?
- How many tests will there be and when should they be given?

I have to do other mundane things such as working out the problems that I assign to my students so that I am aware of possible snags, and so that I have appropriate hints in mind for students that might have trouble.

It is harder to describe what I might call my "shepherding" responsibilities. In order to keep the class running smoothly, I have to play an active role in the pacing of the class. Sometimes it is important to let things move fairly slowly to give the students a chance to assimilate some subtle ideas. At other times, I have to prod my students a little bit to get them through more straightforward ideas in a timely manner. It is not possible to describe this ebb and flow exactly. Different classes may respond differently. Keeping close tabs on the reactions of my students helps me to know how to proceed.

Though I place a great deal of stock in having my students arrive at their own answers to the mathematical questions raised in the course, it is unproductive to let them wrestle forever with any one difficult issue. I have to pay close attention to the students so that I know when they can overcome an obstacle on their own and when an additional hint might be necessary. (Of course, how readily I give a hint will also depend on the specific topic at hand—some things are worth more wrestling than others! There are topics that can be left "hanging" for further student thought while the class moves ahead, whereas other obstacles must be surmounted before any further progress can be made.)

Some Foundations students start the course more able to handle abstraction and rigor than others. Some students adapt very rapidly to the rigorous thinking and precise use of language involved in proving theorems. Others have a great deal of trouble. All students have the opportunity for growth. As much as I can, I tailor Foundations so that individual students can proceed along a path which will ultimately lead them toward mathematical independence. Naturally, students with different levels of aptitude and preparation require very different sorts of stimulation and help from me. As a result, I spend a great deal of time during the semester working individually with students in my office. This can be quite time-consuming but is very rewarding, and it helps me to give individual students exactly the hints or help that they require to proceed on their own.

Two Typical Class Meetings

It is a dreary Wednesday afternoon in the last week of February. This is more than one-third of the way through our semester, so the Foundations class is well underway. The students by now understand what is expected of them and things are running smoothly. The quality of the class presentations has been steadily improving. We have been working our way through the section on orderings (4.2) for the last couple of class meetings, but we have yet to tackle the least upper bound.

I never begin a discussion by stating a definition. The students in the class are supposed to have read the definition already and worked through the associated exercises and examples. Nevertheless, I like to initiate a discussion that will naturally lead to a review of the definition.

I decide to begin by going over student answers to Exercise 4.2.21. This exercise asks the students to interpret the definitions for upper bound, least upper bound, and so on in terms of the ordered set they understand best: $(\mathbb{R}, \leq)$. The discussion of this problem takes a few minutes, because in the course of the discussion I ask slight variations of the questions that are written and manage to bring up most of the issues associated with the relevant definitions.

At this point, I set students to work on problems 9, 10 and 11 at the end of the chapter. (Pages 98 and 99.) They work in small groups of two or three. I circulate around and discuss their progress with them. In these problems the students are asked to explore the notions of upper bound, least upper bound, greatest element, lower bound, greatest lower bound, and least element in less familiar contexts. They are especially chosen to tease out subleties that the students might miss in the more familiar setting of the real numbers. I give the students about 20 minutes. Some students will have completed most of the problems in this time. Others will still be struggling with the first or second. Nevertheless, I stop the work and move to the next idea.

I take my seat and ask for a volunteer to prove Theorem 4.2.22, which establishes the uniqueness of least upper bounds. An average student volunteers to present his proof. His presentation is good, but his proof is problematic.[4] He proceeds by contradiction: Suppose there are two distinct least upper bounds a and b. Then either $a > b$ or $a < b$. In the first case, a cannot be the *least* upper bound, and in the second case b cannot be the *least* upper bound. So there can only be one least upper bound.

The other students spot one of the flaws at once. Since we only assume that we have a *partial* ordering, we do not immediately know that a and b are comparable. Once this is agreed upon, I ask the question, "Is this proof valid in a totally ordered set?" Most of the students agree that it is.

I take the opportunity to remind them of the standard technique of approaching uniqueness proofs, first outlined in Chapter 2: assume there are two and show they are equal. The volunteer tried a different tack: assume there are two that are different and derive a contradiction. This would be correct logically, but the standard approach is most often easier and more direct. As usual when a student presents an erroneous proof, the problem is now "his." He will have an opportunity to present a revised proof at our next class meeting on Friday.

I finish the class by telling the students to finish their work on Problems 9, 10, and 11, if they have not already done so. They need not turn in their solutions, but I suggest that a good understanding of these problems will serve them well on the upcoming midterm. I will be glad to discuss any difficulties with these problems during my office hours.[5] I also take the opportunity to talk about the meaning of the word "lemma" and tell the class that they should work on proofs for Lemma 4.2.25 and Theorem 4.2.26—these will be presented during the next class period.

Friday's class . . .

The first part of the class is taken up by a new attempt to present the proof of Theorem 4.2.22. This time the student gives a proof, which after some minor suggestions from classmates is pronounced correct.

When that is concluded, we briefly discuss the least upper bound property and its importance (especially) in the real number system. We also go over the significance of problem 11(c) (p.99). Then I ask for a volunteer to present Lemma 4.2.25. One of the best students in the class agrees to present the lemma and does a very good job. Another good student suggests that a sentence should be added in order to justify a particular claim, and this is done. I can see that, despite the silence, other students in the class are reeling from just trying to understand what is going on, so I let everyone ponder the proof for three

[4]Actually, I have never actually seen this particular mistake, but it is a good illustration of what happens when an erroneous proof is presented—which is something that can occur in a "typical" class!

[5]Students who are having trouble with these problems are having trouble with the definitions and *need* to come talk to me!

minutes, on the clock. Everyone claims to be satisfied; my guess is that most of them are, but I can see that a few of the weaker students are "hiding." I may find an excuse later to talk to them about how things are going. Perhaps I can get them to come in for individual help.

We move ahead, I ask for a volunteer to present Theorem 4.2.26, but I don't really expect one. To my surprise, a student raises her hand. (The same student that made the suggestion for adding a sentence to the proof of the lemma.) I am surprised, but delighted, because I have never before had a student volunteer to present the proof of this theorem. I have always ended up assigning it to the class as a written assignment.[6] The presenter does a splendid job. I guess I will have to think of another major written assignment for this class.

There are about fifteen minutes left in our class period. This is just about the right amount of time to go over the basic notions of "pairwise disjoint" and "partition," as a prelude to our study of equivalence relations. I give a little preliminary introduction. Then I lead the students through the definition of pairwise disjoint; we construct several examples as a group. I get the students discussing the notion of "mutually exclusive and exhaustive categories." As we discuss the idea, we gradually zero in on the definition of partition that is given on page 79. We finish the class by working through Exercise 4.3.5 which asks the students to construct various specific examples of partitions. I end the class by telling the students to carefully read through this information to set it in their minds and then to work through the ideas discussed on pages 80 and 81. We will discuss these ideas and present the proof of Theorem 4.3.8 on Monday.

Testing and Grades

This semester I am teaching Foundations; I have divided up the grade as follows:

Class Work	45 %
In-class Midterm	5 %
2 Take-home Midterms	15 % each
Take-home Final Exam	15 %
In-class Final	5 %
Total	100 %

[6] This really happened in my class last week. In the first edition of *Chapter Zero*, I did not state Lemma 4.2.25 separately. I gave a hint that pointed toward it, but the hint seemed to confuse students more than to help them. When I was working on the second edition, I decided that the explicit statement of a lemma would help. The two students who volunteered were among the best students in the class, and my guess is that the weaker students will still struggle with this proof. Nevertheless, the lemma seems to have helped make the problem more tractable for at least some students.

I have tried other, more complex schemes, including giving a portion of the grade for writing style—the criteria applied being correct use of language, and clear and concise expression. (The grade was given based on the progress made by the student over the course of the semester.) I also have given frequent pop quizzes whose major aim was to make sure the students could give basic definitions and examples. Such quizzes simply monitered whether everyone in the class was doing the reading. So my grading system changes somewhat from semester to semester. However, I have come to prefer a fairly uncluttered grading scheme such as the one given above.

A large portion (45%) of the grade is given for the work that students do day to day. "Class Work" includes written assignments, class participation, and in-class presentations. All students are expected to participate regularly in class discussion and to put in their fair share of time at the board. Since students cannot participate if they are not present, class attendance is mandatory. I don't make the "teeth" on this policy explicit, but missing classes without good reason is understood to count against the student's participation grade at the end of the semester.

As you see, I give two exams during the semester. Each of these exams has both an in-class and a take-home portion. The in-class portion is worth much less than the take-home portion.

I do not ask the students to prove theorems on the in-class exam. Instead, the in-class portion is meant to be a straightforward, objective test that measures how well students have the "facts" at their fingertips. I always ask the students to give several definitions. The rest of the test consists of true/false or short answer questions. These often (but not always) require a short justification. Among the short answer questions I usually include some that ask the students to give examples of objects we have studied. If the students have kept up with the readings and discussions, putting the ideas together in their minds, they should be able to do very well on this portion of the exam.

The take-home portion of the exam consists entirely of proofs that the students haven't seen before—usually 6-8 problems, several of which have multiple parts. I try to include a wide range of difficulty in these proofs. I give a couple of easy and short arguments that just test the students' ability to "follow their noses" through the logic from the definitions to the desired conclusion. Each exam also has a more difficult proof that requires a real idea or a deeper understanding of ideas covered in the text (or both). Certainly such a proof will have several steps so that the students will have to sustain a chain of reasoning in order to achieve their goal. Most of the problems lie somewhere in between. Sometimes I think up problems and sometimes I steal them from books. In addition, I have some general sorts of things that I like to include on exams.

- Many times I give harder problems by breaking them up into multiple parts that help the students find their way through the ideas and also give them a good chance for partial credit. This makes such problems more tractable and puts them in the middle level of difficulty.

- I almost always have a problem in which students are asked to decide whether a certain mathematical statement is true and to justify their conclusion by giving a proof or a counterexample.

- Very often I introduce a new idea by giving the students a definition and leading them through some simple propositions.

Time frame: Students frequently must balance the competing demands made by their various classes. I therefore used to give my students a week to work on their take-home exam, thinking that it would not be so hard for them to find a couple of days within that period to dedicate to the work. However, I discovered that some students were spending most of the week working on it, while others (who were unlucky enough to have other papers due or other tests to study for) could only use a day or two of the time. I tried making it just a two-day turnover, but then some unlucky students got to spend very little time on it. It never seemed to work out fairly. I have at last arrived at a compromise scheme. I seal each test in a manila envelope with places marked "time opened" and "time sealed" on the outside. Each student can then pick any 48 hour time period during a specific one-week span to work on the exam. The students are on their honor to work only for the 48 hours.[7]

[7] I suppose there are students who abuse this privilege, but I already place so much trust in my students by giving them a take-home, that I reckon that worrying about it is like "swallowing the camel and straining on the gnat."

Strategies

The First Few Class Periods.

Why I Don't Lecture

I can think of lots of good reasons to start the Foundations course by lecturing.

- It avoids wasting that first day of class, when students have not yet read anything or done any problems.
- It is a fast way to get through the highlights of the first two chapters. This way I can make sure the important points are clear without spending too much time.
- The content in the chapter on Logic is extremely important and if I just get students to present problems, many of the important ideas may be overlooked. (This is seldom the case in later chapters.)
- Since it is what they are used to, lecturing makes the students feel more comfortable in the first few days of classes . The material is very different from their other math classes, but at least the format is the same.

Well, . . . I used to think that making the students feel comfortable was a good reason to lecture. I have changed my mind. Students come to my Foundations class with certain expectations about what sorts of things I will do and how I will do them. They have equally strong assumptions about what I will expect from them and what they will need to do to fulfill those expectations. These are reasonable assumptions based on their experiences in previous math courses, but they are pretty much all wrong. Foundations is conducted very differently from any other math course they have ever had. No matter what I do, it is going to take a while for them to understand what I want from them. I have learned the hard way that it is very important to burst the bubble of their expectations right from the start. If I behave "normally" by lecturing in the first few days, many students will cling to that normality, even when the class routine moves away from it. The fact that lectures make the students feel comfortable is the best reason to avoid them in the first days of class.

There are other reasons for not lecturing in the first few class periods.

- Most students could pick up 75 % of the chapter on Logic by simply reading carefully, but if I lecture they sit passively taking notes. In fact, many of the students in Foundations have never actually read a math textbook, relying instead on class lectures to tell them what they need to know. The chapter on Logic is a good chapter with which to begin learning to read mathematics.

- When I have lectured at the outset, it has taken me much longer to get students seriously working through the material outside of class. When they finally buckle down to do it, it is on harder material, and this can be very frustrating for them.

- Many students firmly believe that they understand material better from lectures than they do from working through it on their own. I firmly believe the opposite.

  It is true that working through material is harder than listening to a lecture on it. But lectures are polished and smooth. They make everything seem so straightforward. Students may then *feel* in control of the material when they really aren't. If students do not begin by grappling with the easy stuff, they may find themselves overmatched later when the material is harder. A slight vagueness in their understanding of logic and sets can very rapidly grow into a major problem.

In my view, the negative consequences of lecturing in the first few class periods far outweigh the advantages. Thus, I steadfastly avoid lecturing for the first few weeks of classes. An isolated lecture halfway through the semester seems to be relatively harmless, however, and is sometimes useful as a time-saver. The students find it a welcome relief but know that it is a temporary expedient; they have no trouble going back to the usual routine after the lecture is over.

New Habits, New Routines

The reason that I choose not to lecture is a negative one. That is, I *don't* want the students to think I am going to behave in certain ways, and I *don't* want them to slip into old patterns. However, the old behavior patterns will creep in anyway if new behavior patterns don't take their place.

On the very first day of class, I give my students a written sheet outlining their responsibilities for the course, and I go over it. This helps get the message across, but it really doesn't thoroughly sink in. The students do not consciously think that I am unserious about what I claim to expect; they just don't have any idea what I mean. I thus have to *show* them what I have in mind. Here are some likely scenarios.

- Suppose that the first time I give the students a reading assignment, they say "I didn't understand any of this." (This is rarely entirely true, but learning to read the book does take some practice and persistance.) I have to resist the temptation to respond by immediately going over the material. If I do, the students will get the message that if they claim not to understand, I will simply accept that and lecture.[8] What I do instead is to help them step through the ideas—requiring a lot of input from them. Basically, I show them some of the tricks I use myself to figure out something new to me. This has the same effect as lecturing on the material—we end up going over the major ideas in class—but it helps the students to see what they need to be doing when they read. It also puts them on the spot a little, so that they learn that they cannot get out of reading the material by feigning ignorance.

- I set very high standards for class presentation. At the same time, I realize that students are not born knowing how to make an oral presentation and most of them have had little or no practice. Part of the early work of the class is to talk about what goes into a good presentation.

  The first few class presentations are likely to be pretty slipshod affairs. Students don't write in complete sentences; they don't worry about punctuation; they write the entire work on the board with their back to the class and without saying anything; they ask (me) if there are any questions as they pick up their book and head back to their seats; and there are probably other transgressions that I can't think of right now. *It really doesn't work to let this pass, hoping that over time the students will improve.* On the contrary, my experience is that, if I don't say anything, things just get worse.

  As a student writes, I try to make quiet and friendly suggestions—like "why don't you explain what you are doing as you go along?" or "you are saying lots of words that you are not writing on the board. Why don't you try writing them down?" Sometimes the student picks up on my suggestions, but usually she will do what I suggest once or twice and then stop. After the presentation is complete and the mathematical questions are answered, I do a pretty thorough job of picking apart the presentation. I try to be nice. I apologize to the person on the spot, assuring her that I always have to do this to the first few people who volunteer. I explain that this is all part of the learning process and making a good class presentation is not something we are born knowing how to do, but nevertheless, the process is painful for the students and for me. It is painful, but essential. It works. Before long the students are giving pretty darn good class presentations.

- Though I go on a lot about presentations, I emphasize the role of the students who are sitting down by leaving comments about the mathematics almost entirely to them in the first few days. If there is serious disagreement about whether something is right or wrong, I let them toss it back

[8] At some level, even bright and motivated students will absorb this message.

and forth for a while. If they don't resolve the issue, I may give them a hand, or I may tell them to think about it until the next class period instead of resolving it for them.

If I can get students seriously working through material on their own outside of class, giving careful class presentations and participating fully in class discussion from the outset, they will do well individually and the class is guaranteed to go smoothly.

Allaying Students' Fears

My students all know (usually before they sign up for the class) that in Foundations they will be expected to prove theorems. They often start out the class intimidated. Many students (even those who eventually come to thrive on it) are scared to death of the idea of proving theorems on their own. I think that the main reason for this is that the only theorems students have seen proved in their previous math courses were major theorems. Arcane language was used and much machinery was built (seemingly out of thin air) in order to prove the theorems. I spend a lot of time in the first few weeks just reassuring my students that I won't begin by asking them to prove things like the Mean Value Theorem. I have to convince them that the tasks I am assigning them are doable.

The reassurances I give are various. The idea that we start out with things that are fairly easy to prove is a standard. There are other things that students need to be reassured about. For instance, a common student myth is that if one can get the idea of how a proof should go, then writing it down should be a triviality. I make it very clear that learning mathematical language is not an aside, that it is a central theme of the course. I add that using it takes some getting used to and that we will be concentrating on use of language a lot in the first few weeks. Mostly this reassures the students because they know that they are not the only ones having trouble with it, that I recognize that it is a problem, and that solving it will be a gradual learning process with which I intend to help them.

Part of the process of allaying students' fears is to convince them to have a little patience with the whole process, and to point out when progress has been made. An example may serve to illustrate. When my students are struggling with their first element arguments in chapter 2, I tell them that they must wait and see. "In a few weeks you will encounter a problem and think, 'Oh, this is just an element argument!' " Several weeks later, when someone is presenting an element argument and everyone is nodding in agreement, I remind them of what I had earlier predicted. Many faces brighten, because even those who are struggling with the more advanced things realize that at least some things are getting easier. This gives them a feeling of accomplishment and a sense that their hard work is paying off.

At Kenyon the students who take Foundations are a self-selected group. They are mostly people who are seriously considering a math major. Most of them have done pretty well in their previous math courses. Some hit their stride in Foundations and breeze through it. They are in their element. But other students run into roadblocks for the very first time. Even those who have the ability to surmount the obstacles with some effort can become discouraged. I find that these discouraged students often respond well if I simply confess that the course can be hard going, but add that they *can* meet the challenge if they work at it.

What About the "Obvious"?

Along the way, I often encounter students that think that everything we are proving is "obvious" and at root believe that what we are doing is a waste of time. In response to this, I stipulate that many of the results we are proving are unsurprising to us. I say that we prove such results for two reasons:

- Since the students are only just learning to prove theorems, I give them theorems that are fairly easy to prove. Deep results never are. "If you can't prove things that are obvious to you, how do you expect to be able to prove things that are not"?
- I point out that we are building a mathematical framework from which to work. This we have to do from first principles. "If you are not sure that your assumptions are strong enough to allow you to prove the 'obvious,' how do you know you can trust them when they tell you something that isn't"?

The word obvious is a tricky one. Students often use it to describe something they already "knew" from their previous math classes. Calculus students who have had calculus in high school often tell me that the product rule is "obvious!" By "obvious", I eventually want my students to mean, "It is easy to see how this follows from previously established facts or assumptions."

I don't harp on this, but I manage to work it in repeatedly. I emphasize that a natural question arises from the statement "That is obvious": "Why?" I also try to get them to see that "Because it is" is no answer.

Common Traps and Problems

Over the years, I have come to recognize a number of difficulties that can arise in Foundations and interfere with the progress of the class. Some are my problems. Some are problems faced by individual students, while others affect the class as a whole. If attacked early, many of these can be easily fixed or averted. Some are less tractable and require constant attention. For all of them, forewarned is forearmed.

How Much Do I Tell Them?

I used to be very bad about giving too much away. When a student asked a question, I was too apt to just come out with the answer. However, one of my goals is to help my students find the answers on their own. I have worked to find ways of being helpful without saying too much.

I converse more freely on specifics of the material with single students or pairs than I do with larger groups. The reason for this is that I can gauge better exactly where the difficulty lies and just how much I should say. If Carlos is struggling with a concept, I can help him work out an example. If a Peter and Susan are stuck on a proof, I try to give them only just enough information to get them unstuck.

If Monique claims she is stuck, I begin to ask questions. If it seems to me that she has not really thought long enough to merit a hint, I might say, "I think you can get this. Work on it some more. Go back to the definitions, think through the ideas one step at a time. Try to answer this list of questions for yourself. If you are still stuck in X come back and see me." (Where X is an appropriate amount of time.)

How Do We Think About This?

An important aspect of one-on-one interaction with students is that I have the opportunity to guide the *way* in which they think through problems. It used to amaze me how hard it was to get some students (especially early in the semester) to do something as elementary as to start by going back to the definitions. I ask Marcus (who claims to be completely stuck on a proof) for the definition of the main term he is working with, and he can't give it to me. If I simply make him go back and read the definition aloud to me and relate it to the problem at hand, he often figures out what he needs to know right on the spot. After this has happened a couple of times, Marcus begins to see that this should be his first step in analyzing a problem.

There are, of course, other simple strategies that we all use to get a handle on a problem. Lindsay's difficulty is that she tends to sit deadlocked, staring at a blank piece of paper, hoping the proof of a theorem will just pop into her head. Lindsay must learn to ask herself simple questions that will help get her ideas flowing, but this doesn't come naturally to her. As I work with Lindsay, I ask her the sorts of questions that I would ask myself if I were in her situation. When she can answer these questions, she is well on the way to proving the theorem. More importantly, she learns something about how to formulate such questions for herself.

Definitions: I do not remember ever struggling with this as a student, but I have come to understand that there is an often ignored, but fundamental, pedagogical issue associated with definitions. Mathematicians think of definitions in very different terms than do most other people. In other contexts, students have used definitions only to "get the general idea" behind a notion. One reads a definition, sees how it is to be used in a sentence, and that is that. No one in his or her right might would *memorize* a dictionary definition word for word! Students have never had to use definitions in the precise and detailed way that we do. They basically don't believe that they have to pay attention to what they think of as "the minutia." They figure that if they can give an example of a partially ordered set and an example of a totally ordered set, then they no longer need to think about the precise wording of the definition at all. We have to *train* them to think of the definitions the way we do. We have to teach them *why* it is necessary to pay attention to the precise wording and *how* to use mathematical definitions as tools that drive mathematical argument and discourse.

Too Many Symbols, Not Enough Words

Here is a typical early proof presentation from Patrick, a student in Foundations.

Patrick says:	**He writes on the board:**
We want to show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
Suppose that x is in $A \cup (B \cap C)$.	$x \in A \cup (B \cap C)$.
That means that either x is in A or x is in $B \cap C$.	$x \in A$ or $x \in B \cap C$.
If x is in $B \cap C$ then x must be in both B and C.	$x \in B$ and $x \in C$.

. . . and so on . . .

Patrick's proof and the oral part of his presentation are pretty good. But what he is writing on the board is not by itself sufficient, or even correct. He has left out the words which convey the logical relations between his formulas.

This habit is quite common among math students, many of whom have unfortunately absorbed the idea that only formulas are important in mathematics. In Foundations (and beyond) it results in three problems. First, if the notes taken by the other students are as sketchy as what Patrick has written, they will not be able to make heads or tails of them later, even if they understand the proof at the time when Patrick explains it. Second, the students are often one or two lines behind Patrick as he gives his presentation, and won't catch the verbal explanations he is giving as he goes along. Third, what is a little sloppy in a simple proof becomes utterly incomprehensible in a more complex proof. The connectives and quantifiers are not ornamental: they are exactly what converts a sequence of mathematical assertions into a proof.

I jump up and down and earnestly wave my arms, and eventually my students get the idea that words are important, at least to me. In the second phase of this problem, many students will liberally sprinkle "thus" and "however" throughout their proofs much as one might sprinkle salt on food to make it tastier. At this point I have a little talk with them.

I tell them that the English text in a proof should do two jobs. First of all, it should clearly spell out the logical structure of the argument. That is, the text is required for *correctness.* Second, the text is also required for *clarity.* I say something like,

> Your task as a writer is to give the right cues to your readers, cues that will make it as easy as possible for them to understand what you are trying to say. Consider for instance the phrases "therefore", "by hypothesis", and "it follows from what was previously shown". Each of these means that what comes next follows logically from previously assumed or deduced statements. But they indicate different things to the reader. "Therefore" suggests that what you say next follows from what you just said. "By hypothesis" means that you are appealing to an explicitly made assumption. "It follows from what was previously shown" indicates that the reader will have to go back some distance in the proof to find the grounds for your next statement. (To make the reader's job even easier, you might say "as I have shown before" and then re-state the previous conclusion that you are about to use.)[9]

This all seems to make sense to students, but only after they have made their own attempts to write and decipher mathematical language. The talk is most effective if it follows a presentation in which connectives seem to be used only as "seasoning" and in fact do not convey the correct logical relations. I just have to look for the right opportunity.

[9]Students at this stage will typically use some variant of "therefore" in every one of these instances. Thus my little talk also serves as a suggestion that variety of usage does more than just keep the reader from getting bored.

Writing clearly is also the best guide to determine the level of detail appropriate to a particular proof. Too much detail can obscure the message almost as surely as too little detail. This balance is a much trickier thing to gauge and students develop their instincts for this more slowly. Since students typically want to write too little, at this stage, I tell them that if they are in doubt they should err on the side of writing too much.

Winging It

Marsha does reasonably well on her written assignments, but when she presents her work in class she is often awkward, disorganized, and hard to follow. The root of her problem is that she thinks that preparing to make an oral presentation is *easier* than putting together a written assignment. Once she has the general idea of a proof, she makes a few sketchy notes and figures that she is ready.

But when she gets to the board, it becomes clear that she is not. Her notation is often inadequate or confusing. She may hesitate between one step and the next because she has forgotten (and never wrote down) her justification for the step. She does not give her fellow students a clear idea of where she is going or why. Paradoxically, Marsha may be far too reliant on her notes, sketchy as they are, because she has not reviewed the argument thoroughly enough to recall its essential structure. She may even discover a mistake in her proof that had escaped her notice before because she did not bother to write out the details. If Marsha had just written her proof carefully beforehand and spent some time thinking about it, most of these problems would not have occurred.

Unfortunately, most of the students in the Foundations class start out with Marsha's problem. They believe that an oral presentation is only half the work of a written assignment, but the reverse is more nearly true. It takes a while to convince them of this. I try to establish high standards for oral presentations, and I do this by making pointed suggestions for improvement in specific instances, particularly in the early part of the course. Students in Foundations know from the outset that my practice is to grade them on both mathematical content and quality of presentation.

Putting Off the Inevitable

Jim, one of the weaker students in Foundations, is scared to death. He knows that eventually he will have to take his turn at the board, but he figures he isn't ready yet. So he sits quietly in class and watches his fellow students. To make matters worse, because Jim is feeling lost he is reluctant to ask questions or participate in the discussion. He is probably not having a lot of success with the problems outside of class, either. As his questions go unanswered, things just get worse.

The problems and theorems in the later part of the course are not as easy as the ones near the beginning; thus, as Jim procrastinates, the most tractable problems pass him by. He feels less and less ready to make his debut at the board. Later, when it becomes critical to his grade for Jim to start making presentations, he will be facing harder material with less experience.

If several people in the class share Jim's problem, by the middle of the term the class will have split into two groups: one group is actively participating and learning a great deal, but the other is getting less and less out of the course with each passing day.

I have found that this sort of thing requires early intervention. From the very beginning of the course, I call upon individuals to give their solutions to exercises and examples as part of the class discussion. (Exercises and examples are supposed to be done by everyone as they read the material, so I do not feel reluctant to "pick on" individuals by name.) I also try to identify the students in Jim's situation within the first two or three weeks of class. Assigning them specific problems that will get them before the class early can help get them past the initial barrier. I may also encourage them to speak to me outside of class. Anything that can "get them going" will make a great deal of difference to their ability to succeed in the class, but they need continual encouragement and incentives to keep at it.

Passive Student Syndrome

Alice really liked calculus, and she was pretty good at it. But she doesn't enjoy Foundations. She is not "in love" with proofs and theorems and does not really see the point in them. Consequently, Alice won't work on any problem that is not specifically assigned to her and will never volunteer for anything in class. To be fair, she is fairly diligent about the problems she cannot avoid and does not skip class meetings. But she seldom participates in class by asking questions or making comments.

Alice finds the student presentations completely unhelpful. She thinks that the class time is mostly wasted. She believes that she would learn more in Foundations if the professor would do a lot more lecturing.

Alice has strong expectations about the way things should be done, and those expectations are not being met. All of her previous experiences in mathematics classes have prepared her to function by responding to things that are taught *to* her. The professor lectures or the book explains, and then she is ready to apply her knowledge. She does not at root believe that she can start her work *before* she understands the subject, that gaining understanding is in fact part of the work. She lacks initiative in part because she is waiting for the professor to get her going.

Basically, Alice doesn't get it. She does not understand or appreciate the purpose of the Foundations class. She may secretly believe that the seminar-style format of the class has been chosen because it is *easier* for the professor! As a consequence, Alice does not participate fully in the class and doesn't get much out of it.

Changing Alice's attitudes is a difficult task. The only way I have found to deal with Alice is to try continually to educate her about the purposes of the course and its format. This is basically a selling job. "Alice" (who is a double major in math and economics) came in to my office last spring and said, "With all due respect, what is all of this for?" I had the presence of mind to pull Gerard Debreu's *Theory of Value* from my shelf. Debreu won the 1983 Nobel prize for his investigations of the mathematical foundations of economic theory. *Theory of Value* is a great little book that illustrates how abstract applied math can be. I opened it at random to a page that was brimful of theorems and proofs about partial orderings, which we had just covered in Foundations. I am not sure if I sold Alice on the ideas of the course, but it made her a little more patient.

Who is he talking to, *really*?

This is not really Bob's problem, but I notice it while he is presenting a proof at the board. I notice three things:

1. Bob is making eye contact only with me. In fact, he is clearly talking to me rather than to his fellow students. He is on the spot, and believes that his job is to demonstrate to me that he has done this problem correctly.

2. The other students in the class sit and dutifully take notes on Bob's presentation. As soon as Bob is finished, every head in the room swivels toward me, and expectant faces wait for me to pass judgment on the work. The presentation itself was not for them. At best, it has acted as a sort of "answer key" for a problem that they did not themselves get.

3. Worst of all, in the absence of student response, I find myself reinforcing their lack of involvement by announcing whether Bob's proof is correct or not.

The class has fallen into the worst sort of trap. The pedagogical "engine" that drives the Foundations course is the interaction among the students. Bob is not a seasoned lecturer, and if there is no give and take with the other students his class presentation will not transmit information very effectively.

It is quite easy for this to happen. In fact, it is the natural inclination of the students (and often mine, as well). In most math classes the primary interaction is between the teacher and the students through polished lectures and occasional questions. The seminar-style format all too readily degenerates into a poor copy of this standard model. I find that I must be on my guard to prevent this dynamic from becoming established in the first place, because once it is established it is extremely hard to eradicate.

On the one hand, this is an easy trap to avoid because if I simply refuse to cooperate, Bob will be forced to look to his fellow students for feedback. "Don't look at me," I tell him. "I already understand this stuff. Ask them if they do." When Bob turns to the other students, they must actively engage his work in order to evaluate it.

On the other hand, I find that I must work to stay impassive. My students rapidly learn to read my facial expressions and body language. I am not exactly a poker-face by nature! Actually, when I taught Foundations a few years ago, I may have gone overboard in my effort to avoid giving anything away. Word got back to me that some students thought I always looked bored. Actually, the reverse was true. Probably the more bored I looked, the more interested I was, and the harder I had to work to contain my reactions.

Misplaced student solidarity.

When Karen presents a problem at the board, the other students hesitate to ask any questions or make any comments about her presentation. They think that this would amount to a personal attack. Ironically, this is far more terrible for Karen, whose work is met with stony silence and downcast eyes. (As a teacher, I know of no more disheartening reaction than that!) This is perhaps the hardest problem to fight. Students think they are doing Karen a favor by keeping silent. Worse, Karen may agree, at first. Her response to her ordeal may be, "Just think how much worse it would have been if they had spoken up!" I tell the students, quite frankly, that by not reacting in any way they are leaving Karen quite alone at a time when she feels most vulnerable and most desperately needs to feel that she is part of the group.

I repeatedly stress the fact that the work of the class is a cooperative effort. The students sitting down not only help themselves, but they help Karen by asking questions and making comments that allow her to make the clearest possible presentation of a correct proof.[10]

I also stress that there are ways of asking questions and making comments that are polite, friendly, and non-threatening.

> *I am a little confused about how you got from the third to the fourth sentence. Can you clarify that for me?*

is much nicer than

> *I don't think you justified that very well. I think you ought to explain it better.*

When a particular point is confusing and Karen is having trouble clearing it up, I often open the floor to the whole class. The students who understand this point all try to think of wording that will make it clearer; this helps everyone and takes the "spotlight" off of Karen. Furthermore, when Karen makes a good, clear and correct presentation, I tell students that they should feel free,

[10] One thing that helps at this point is that if Karen's proof is actually in error, she is penalized in no way. On the contrary, the problem is now "hers." She is given until the next class period to correct her error (consulting with me, if necessary) and then another opportunity to present her corrected proof. I keep no record of the false start. Since this is so, students don't feel as reluctant to point out an error if it is present.

indeed obligated, to tell her that she has done a nice job and that they have no questions. Stony silence is no fun here, either.

Once students start making comments, it doesn't take long before they realize that it is helpful and more fun for everyone (including whomever is at the board) if a lively discussion accompanies class presentations. This makes the class proceed a lot more smoothly and productively.

I have heard a suggestion that I think is great but my own temperament prevents me from pulling it off very successfully. A rule can be made that allows the instructor to ask any question she likes, but never of the student who is at the board! She may only ask questions of the students who are sitting down. Students in the class who ask (substantive) questions of the person at the board are exempt from being asked a question by the instructor. This tactic can be used to get a discussion going if there are questions silently hanging in the air. (The best thing is that if the instructor carries this off successfully a few times, the students will ask questions to preempt the possibility of being asked questions themselves—thus voluntarily doing what one wants them to do anyway.)

Becalmed

No matter what I do, there are times when the work of the class slows and the students aren't solving problems fast enough to fill up the class period. This seems to stem from a combination of two factors: the work in Foundations is getting harder, and the work in other classes is piling up. The first of these factors slows progress despite the students' best efforts. As for the second, since most presentations in Foundations are done by volunteers, the paper that is due tomorrow in political science takes precendence over the work (that they can leave to someone else) for the Foundations class. "I will volunteer next week." The problem with this, is that English classes, history classes, sociology classes, and art history classes all over campus have papers due that same week. Too many students end up thinking that someone else will be there to volunteer.

Attempting to wait out the situation (thinking "They will have gone much further by the next class period") seems to give the disastrous signal that it is alright if a class period or two go by without any student having a problem to present. However, I have found a simple way to get things going again. I divide the class into small groups and assign specific problems to the groups. Somehow, when fellow students are depending on them, students find the hours and the ability to get the work done, and in good time.

This strategy has never failed to get things going again. After the groups have presented their work, I go back to the volunteer system. The transition back has never posed any problems.

Group Work

When the class gets "becalmed" or when we are studying a section that contains many problems, it is useful to divide the class into small work groups (of 3-4 students), each of which is responsible for presenting specific problems to the class. The problems assigned to a single group will not be consecutive problems. I make the assignments in such a way that the various groups will take turns presenting their work. Thus each group must be prepared to present one or two things during each class.

How I Constitute the Groups

- I try put people who are already work regularly (and effectively) together into the same group.
- I try to match the abilities of students that are working together. I have found that if a group has a strong student and a couple of weak students in it, the strong student will have solved the problem before the weak students have a chance to thoroughly think it through. Then the tendency will be for the student who has worked out the ideas to simply "fill the other students in." They will not get nearly as much out of this as they would if they had been in on the original thinking.
- If I have identified any students who don't work very hard, I try to put them together. They may sink each other, or they may start working since they are on the spot and have no alternative!

How I Assign Work to the Groups

- I try to make sure each group is working on a varied enough collection of arguments that the members of the group will have to understand all of the major ideas in the section. This prevents students from just ignoring an important definition. (Though each group will only be presenting specific problems, I tell the students that they are still responsible for reading all the material—which includes doing the examples and exercises—and understanding the definitions and the statements of the theorems. Nevertheless, I suspect that students don't pay as much attention to ideas on which they are not going to be immediately and thoroughly grilled!)
- I try to rougly match the difficulty of the assigned problems to the abilities of the students in the group. The weaker students thus work on problems that are tractable for them, and the stronger students get problems that give them a bit of a challenge.

Class Logistics

- The various members of the each group take turns being the presenter of the group work. I usually assign sufficiently many problems so that each member of the group will have to present something. This assures that every person in the class will present a proof during these few days.
- We go through the material in the book as we ordinarily would, but instead of asking for volunteers, I just call on the group that is responsible for presenting a particular proof.
- In order to make sure that students are carefully following the work of other groups and participating in their presentations as usual, I sometimes make a written assignment in which students must write up (in their own words) proofs of theorems presented by people in other groups. (I tell them ahead of time that such an assignment is coming, but I don't specify the particular problems until after the presentations are finished.) This is a pretty straightforward assignment for them, but it keeps them honest. It is easy to tell when students are just parroting, without understanding, what was presented in class.

Things move quickly and smoothly. The quality of the work is at least as good if not better than it is when we are working entirely with volunteers. There seems to be no problem going back to the volunteer system when the group assignments end.

Results

We started teaching Foundations at Kenyon about a dozen years ago. Since then, the mathematical maturity of our junior and senior math majors has noticeably improved. Furthermore, now that we can assume a certain core of knowledge and experience in our students, our upper-level theoretical courses "get off the ground" more quickly and thus go further. As an added bonus, this course has helped us recruit some very bright students for the math major.

Part II
Chapter by Chapter

Chapter 0—Introductory Essay

This chapter sets the stage for the rest of the book, but no later chapter depends on it in any specific way. Nevertheless, I like to have my students read it because it lets them know right from the start that the "game" has changed—that this course will be very different from their previous math courses.

The chapter is not difficult and should only take 15–20 minutes to read. We do not spend a great deal of time on it in class. I let the students' own impressions and questions fuel the class discussion. This is a good, non-threatening way for students to become immediately involved, and it establishes the notion that *they* will be the driving forces in the class.

Saving time

Chapter 0 can safely be skipped if time does not permit you to cover it. I have sometimes skipped it myself. Alternatively, I have instructed the students (by electronic mail) to read it before the first class meeting, as a basis for a class discussion. However, some students understandably think that giving homework before the first class period is a dirty trick.

Chapter 1—Logic

As I've said before, I want to avoid lecturing early in the course. This is especially true on the first day. I have experimented over the years with several strategies for productively using the first day without lecturing.[11] When I was preparing the second edition of the book, I decided to include a set of questions at the beginning of the chapter on Logic. The students would start out by working to determine which of the questions were true, which were false, and by trying to prove their answers. The result was Section 1.1. I think of this as a "thought experiment." Students are tapping into their intuitive understanding of what it means to be true or false, and what it means to *prove* that something

[11]Including, as I said above, giving a reading assignment before the first day of classes. I also tried using the computer game Minesweeper to talk about strategies for proof. (See "Introducing Proof Techniques Using the Logical Game Mine Hunter," PRIMUS, June 1995.) This worked pretty well, but I decided that I needed to give it more than one day to make it a worthwhile experience, and my priorities took me elsewhere.

is true or false. Some of the questions that the students work on are discussed again later in the book, many in Chapter 1. I have had the opportunity to use this activity only once, a few weeks ago, but on that occasion it worked well. The students seemed to enjoy it, and they were actively engaged in doing mathematics on the first day of the course.

The chapter on logic is tricky in a number of ways. On the one hand, the material in it is crucial; if the students don't get a good grasp of it, they will soon encounter difficulties. On the other hand, the students are studying only logical principles and are not really being trained in their use at this time.[12] Thus it is very important to go through the ideas in the chapter, but it is counterproductive to spend too much time on them at the outset. I find that it is much more effective to reinforce repeatedly the ideas of Chapter 1 when theorems are being proved later on. (See the tips at the end of this section.)

I also want to keep the time spent in Chapter 1 to a minimum, so I compromise by combining class discussion with small group work in class. Chapter 1 is well suited for this. I have found that a fluid discussion that uses certain key exercises as "jumping-off points" is effective and fairly rapid, and at the same time gets the students involved. Here are exercises that I have found especially useful for bringing out the key ideas. Students come to class on the second day having read through the first part of Chapter 1. The first few pages are fairly self-explanatory. I take questions if students have them, but rapidly go on to Section 1.3 if there aren't any.

- Since quantifiers and quantification are a bit subtle, I lead a discussion about them centered on exercises 1.3.1, 1.3.2, and 1.3.3. In the course of discussing these problems, most of the relevant issues naturally arise and can be clarified.

- Though implication is dealt with quite thoroughly and explicitly in the text, it is so crucial that I go over the highlights and address specific issues that are confusing to students. I begin with a short look at Example 1.5.1. Though it is fully written out in the text, and students rarely have trouble understanding it, it is a good place to start. This makes sure that the conversation starts at a place where the students are on solid ground. I move on to Exercise 1.5.2. Usually I can get a student to volunteer to present a solution to this problem. The ensuing discussion inevitably leads to talk about hypotheses and conclusions, counterexamples, truth and falsehood of implications, and even vacuously true statements.

[12] As author, I considered whether to include lots of "practice" theorems for the students to prove along the way. There are a few, but I decided to keep the number to a minimum for two reasons. First, the chapter on logic would take longer to get through than I (as a teacher) am willing to give it. I want to get on to the mathematics! Second (and more importantly), since no framework of assumptions has yet been built—no mathematical content discussed—the students end up assuming things that are no more well-established than the things they are proving. It is important for students at this level to learn that when proving theorems they cannot assume just any mathematics that they have heard about along the way. What they are allowed to assume and what they are not allowed to assume should be fairly clearly spelled out for them.

It can be useful to stress the fact that implication in mathematics is not a "cause and effect" relationship. This is mentioned in connection with vacuously true statements, but can be observed in true statements, as well. Consider, for instance, statements like "If Picasso was a painter, then most cats have four legs."

- Students don't seem to have much trouble picking up how to construct truth tables. I may work one out at the board with a lot of help from the students, but no more than that.[13] Sometimes I just assign one or two for the students to work out and turn in. Virtually all students do well on this assignment, but the homework assignment allows me to catch the one or two in a class that missed something crucial.

  I think it is more important in class to emphasize what may be learned from truth tables. In fact, to make this idea very explicit, I added a section in the second edition that does exactly this. I certainly don't go through all of the truth tables in class, I talk about the interpretation of the statements. I try to show students how to "translate" the symbolic statements into words. To get meaningful statements, they sometimes have to be flexible about the translation. For instance, I would want my students to recognize the following parallels:

Symbolic Expression	English Equivalent
$(A \implies (B \wedge C)) \implies (A \implies B)$	If the truth of A implies the truth of both B and C, then the truth of A implies the truth of B.
$(A \wedge (A \implies B)) \implies B$	If A is true and A implies B, then B is true.

  When I teach Section 1.7, I spend a bit of time going over all the principles illustrated there. It is pretty easy to get the students involved by asking them to participate in the translation process (once they see one or two they get pretty good at it) and by asking them to interpret the information contained in the various examples and exercises.

- The issue of negating statements is very important and students find the details tricky. I make sure to go over the various examples and exercises in the section, but it is pretty easy to do by having students give their solutions to various parts. I usually do this by calling on the students by name.[14] This usually ferrets out individual difficulties that students may be having and it allows me to clarify the main points.

[13] It is also workable, but more time-consuming, to have a student present one.

[14] I have recently started keeping a roster of the class with a little chart that I mark when I call on a student. I tell students that I am not keeping track of whether they answer correctly or not, merely whether I call on them. This allows me to be sure to call on all students more

- The last part of the chapter, Sections 1.9 through 1.14, talk about specific proof techniques. There are some exercises and problems at the end of the chapter that are good vehicles for discussing the relevant issues. Even more than the rest of the sections in this chapter, my general approach is to cover these sections very quickly and to reinforce the ideas later when they come up in context. I think, however, it is useful to try to get the students to articulate in straightforward terms the procedure they will need to follow to implement each of the various proof techniques. For instance, they should ideally be able to articulate the procedure for proof by contradiction, as follows:

  > Assume that the hypothesis is true and that the conclusion is false. Then reason until you arrive at a contradiction.

  They should be able to give similar short synopses for the other proof techniques.

Tips for Reinforcing Logic in Later Work

Here are some good and quick ways that I use to reinforce the ideas of Chapter 1:

- When someone is presenting a proof, I get students to tell me what method of proof is being used. (I don't necessarily ask the person who is presenting the proof.) We briefly review the relevant ideas in context.
- For theorems that are not written in the "If A, then B" form, I make students explicitly state the hypothesis and the conclusion.
- Though I think it is overly pedantic to push this too much early on, it is occasionally worthwhile to reinforce the distinction between proofs by contrapositive and contradiction. (Most students at this stage will give a contrapositive argument and claim it is a proof by contradiction.)
- Students frequently forget quantifiers in their proofs. Point this out when it occurs and have the students notice the ambiguity (or even error) that is introduced as a result. In general, push the correct use of quantifiers. Times where quantifiers need to be negated are also good opportunities to reinforce the concepts in the Chapter 1.
- If I ask the students a question about logic that they cannot answer, I make them (physically and immediately) turn back to the chapter on Logic to look up the answer. This will get them in the habit of looking back themselves when they run into trouble or confusion. I find that I have to be patient. At first students try to "wait me out" certain that I will save them the trouble by giving them the answer.

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or less equally. This is good because it makes the students realize that I am not "picking on them," but it also makes them acutely aware that they may be called on at any time to give an answer. I am very happy with the practice.

- When the first instances of proof by contradiction, proof by contrapositive, uniqueness, existence and so forth show up, I make the students turn back to the chapter and to help them recall what needs to be done. (Even if the student presenting the proof has gone over the ideas and made good use of them, the other students will benefit from having them highlighted and illustrated in the context of the correct proof.) It can be useful to point out specific landmarks, as well. For instance, when a student presents an existence proof, I get other students to identify the section in the proof where a candidate is produced and the section where the candidate is shown to be what is claimed.

Chapter 2—Sets

Once the class gets into Chapter 2, things rapidly settle down into the routine for the course. In this chapter, students encounter their first "real" proofs in the form of element arguments. There are also a fair number of exercises that allow the students to work with set notation and definitions. I have found that it works well to ask the students to present their solutions at the board. There is a wide range of difficulty in the problems that the students are asked to tackle; several have multiple parts. It is an ideal time to encourage every student in the class to get involved. I do this by asking for volunteers in some some situations and by calling on specific students in others.

I always make one or both of the DeMorgan laws a written assignment. This assures that every student in the class has personally slogged through at least one element argument involving general indexing sets.

In the interests of time, I skipped Theorems 2.4.6 and 2.4.11, but I suggested them to students who wanted more practice with the notation and language. Other than that, I covered 2.1 very quickly, and I did pretty much everything else in Sections 2.2–2.4.

> **A Tip:** I think that (in the interests of clarity) it is a good idea to begin by spelling everything out. But after the first couple of element arguments, I start suggesting to students that some things are understood by all and do not need to be spelled out. For instance, they do not need to start each element argument by writing that they must show each set is a subset of the other. They just need to indicate clearly to the reader that each of these parts is taking place.

The power set is a completely new notion to most students, and they will probably need some choice hints and advice, but Section 2.5 seems to come along without too much fuss. I chose not to have the students work through Problem 2.5.7. However, the ideas of that problem are central to initial discussion of induction, so it is necessary to at least give a good intuitive discussion before proceeding to Chapter 3.

I have my students read the section on Russell's paradox and talk about it a bit in class, but I don't dwell on it.

The Rough Spots

Few students experience major mathematical difficulties in Chapter 1, but most run into a few problems in Chapter 2. I think the obstacle is primarily a difficulty with language, not a conceptual one.

The element argument that is written out in detail (ad nauseum)in Example 2.4.2 really helps students get started. Most, using this as a guide, can produce the proof required in Exercise 2.4.4. With a small nudge in the right direction they can also get problem 2.4.8. However, almost all of my students run up against a brick wall when they encounter the more general formulations involving the arbitrary indexing sets.

I have found that two things really help the students to overcome this hurdle. One is just to work with indexing sets, demystifying them a bit. (Exercise 2.3.15 and Problem 4 on page 54 should help with this.) This is a start, but the real stumbling block is that the students don't have the insight to go from language like "$x \in B$ or $x \in C$" to "$x \in B_\alpha$ for some $\alpha \in \Lambda$." The use of the quantifier is not intuitive to them. As a result, the second edition includes, in Example 2.3.12, some explicit hints about the use of language. This helped my students this semester, but they still struggled with the language. I spent a lot of time working with individual students on this, and it took them some time. But in the end, most got it. For students that are having extreme difficulties, I would suggest working directly on a "translation" of the proof of Theorem 2.4.2 into the language of the more general setting. Writing the two arguments side-by-side on a sheet of paper, having the student help you recast each line into the language of general indexing sets can be extremely helpful.

On the principle that being forewarned is being forarmed, one more thing seems worth mentioning. Some students will try to say that:

If $x \notin A_\alpha$ for some $\alpha \in \Lambda$, then $x \in A_\alpha^{\mathcal{C}}$ for all $\alpha \in \Lambda$.

This baffled me at first, but then I realized something. The students that say this view the transition from $x \notin A_\alpha$ to $x \in A_\alpha^{\mathcal{C}}$ as a *negation.* They have absorbed enough to know that when you negate a statement involving a "for some," you should end up with a statement involving a "for all."

Chapter 3—Induction

For the second edition, the chapter on induction has substantially rewritten,[15] and I have placed it earlier in the book. My main goal, however, remained the same. I have noticed that students usually understand the principle of induction and readily learn how to use it to prove simple number theoretic results like the standard sums of consecutive integers and squares and so forth. The trick is imparting the *nuances* of proof by induction. Induction shows up in many contexts and takes on subtly different guises in those various contexts. Different uses of induction may look very similar to a seasoned mathematician, but, in my experience, many students who have seen induction in one context have trouble applying it (or even recognizing it) in others. So I believe it is important to give students the opportunity to see induction at work in different contexts.

The chapter is divided into two major parts. The first part (Section 3.1) talks about what mathematical induction is doing and why it works. The second part (Sections 3.2 and 3.3.) has the students using induction to prove some simple theorems. I have found, first of all, that it is very important to stress the fact that Section 3.1 is *not* talking about how to use induction, but merely about how it works. Otherwise, students try to use the proof of 3.1.2 as a model for using induction. This leads them wildly astray!

To help with this transition, I have started Section 3.2 with an example in which induction is used to prove that a set with n elements has 2^n elements. (This proof relies on Problem 2.5.7, so at least the basic ideas underlying this problem need to be covered before the section on induction.)

After discussing the Principle of Mathematical Induction on one day, I had students work on Problems 3.2.2-3.2.6 in class the next time. In this way, I was able to circulate among them stressing the major issues behind induction and helping the students see how to implement them in a few simple cases. I had my students write up and turn in solutions to Problems 3.2.5 and 3.2.6. In the following class, students worked on Problems 3.2.2 and 3.3.3 which are somewhat more difficult. These problems, I also had them write up and turn in. Though I hesitated in some ways to spend three full days on this chapter, I found that my students really needed this length of time on what is a surprisingly delicate topic.

Chapter 4—Relations

After Chapters 2 and 3, students are at the same time more confident about doing proofs (they managed to survive!) and apprehensive that the next chapter will bring another jump in difficulty equal to the last. It doesn't. In fact, for

[15]The treatment in the first edition really didn't work very well.

a while in any case, things get easier. Checking that relations are reflexive, antisymmetric, etc. comes easily to most students. The work in the chapter gradually gets more difficult, but students seem to handle it pretty well.

Section 4.1—Relations

I cover pretty much all of 4.1, perhaps picking and choosing among the exercises. This is one of those places in which some problems are so easy and self-evident that students will become sloppy about writing things down. The tendency is to write "Reflexive? Yes," and leave it at that. Therefore, when covering the problems in this section, especially 4.1.9 and 4.1.10, I mix oral discussion with board presentation. This makes things go faster and avoids sloppy board presentations. Working through a number of exercises helps to clear up the occasional confusion about reflexivity, symmetry, etc. that can lead to trouble in the rest of the chapter.

Section 4.2—Orderings

There are a lot of important concepts defined in this section. However, at this point in the book, most of them are not taken much beyond understanding the definitions and looking at examples. They really come into their own only later when they are used for something like comparing cardinalities or describing the real number system; thus 4.2 seems a mere hodge-podge to some students. I often just warn them about this from the outset.

From the point of view of the class dynamic, there are really five short but identifiable "stages" in section 4.2. (Each arising from a new set of definitions.)

Stage I Getting comfortable with the initial definitions. (From the beginning of the section through Exercise 4.2.9).

Stage II Lattice diagrams and associated exercises. (Starting with lattice diagrams and going through Example 4.2.10).

Stage III The contrast between maximal and greatest elements. (Definition 4.2.11 through Theorem 4.2.15).

Stage IV Immediate successors and predecessors. (Definition 4.2.16 through Theorem 4.2.18).

Stage V Least upper bounds and the least upper bound property. (The rest of the section).

Stage I—This is the first time that the students have had to work through a serious abstraction of a structure with which they are really familiar. They accept the definition of partially ordered set amazingly placidly. This is understandable since they have just spent a whole section talking about reflexivity, antisymmetry, and transitivity. They feel pretty comfortable with these concepts, and (thinking of the reals) see the rationale in linking these with the symbol $\leq$. However, I remain a bit wary. For some students, the comfort comes entirely from the familiar symbol $\leq$. They think they know *exactly* where they stand: squarely in the real numbers![16] These students really get hung up on the definition of totally ordered set. They cannot imagine what it is trying to tell them; as a result, exercise 4.2.4 ($\mathcal{P}(S)$ is not totally ordered under $\subseteq$), simple as it is, is beyond them. I find it to be worth my time to pause a little more than seems necessary on this exercise to give the students who are baffled some time to catch up. Exercise 4.2.7 is a little harder for the students, but once they have thought their way through that one virtually all will be "with the program." The easy proof of Theorem 4.2.6 is very comforting to students who have been feeling lost.

Stage II— Students have remarkably little trouble with lattice diagrams. Many students enjoy the puzzle associated with classifying all partial orders on sets with four elements. (A number of students enjoy it so much that they insist on working through the entire classification of partial orders on sets with 5 elements!) But the informal look at order isomorphisms can be skipped in the interests of time. If you decide to skip it, however, you will still need to go over the diagrams. They are used to illustrate the differences between maximal and greatest elements, lack of uniqueness of immediate successors, etc. They give the students a *picture* of partially ordered sets that are not totally ordered.

Stages III and IV

Students' intuitions about the differences between partially and totally ordered sets are sharpened considerably by the contrast between maximal and greatest elements (and continue to sharpen throughout the rest of the section). Exercises 4.2.12 and 4.2.13 are excellent introductions. For sharpening students' instincts, I recommend Problems 8 and 10(a-d) at the end of the chapter.

By the way, the examples I have in mind for the third and fourth parts of Problem 8 are an increasing convergent sequence with the limit "on the top" (the ordinal $\omega + 1$) and a sequence of increasing convergent sequences in which the limit of each sequence is the first element of the next (the ordinal ω^2). However, I have gotten a number of very different (and correct) examples from students over the years.

Stage V—Though students will struggle a bit with these definitions, but Exercise 4.2.21 and Problems 9, 10(e,f) and 11 should really help work the kinks out. Theorem 4.2.26 has always given my students problems. The extensive hint that I put in the first edition didn't really seem to help, so for the second edition, I have separated out a lemma that really does seem to help. Two (very good) students in my class this semester, did these problems without talking to me, and did a very good job.

[16]In fact, I have to continually remind some students that the symbol $\leq$ *does not* in general represent the ordinary "less than or equal to" on $\mathbb{R}$ even if we use the same words and symbol to refer to them!

Section 4.3—Equivalence Relations

One often hears math students say: "Equivalence relations are pretty easy—except all that stuff about partitions and equivalence classes. I never understood that." Sadly, the fact that they didn't really get the stuff about partitions doesn't seem to worry them too much. In their view it was secondary, not the main idea. This is, of course, not the case at all; we mathematicians usually think of partitions first when equivalence relations come up. Thus when I set out to talk about equivalence relations, I decided to start by talking about partitions, bring up relations only afterward, and show that a partition naturally gives rise to a reflexive, symmetric and transitive relation—all this before ever defining an equivalence relation.

In my experience, students initially have trouble with the definitions of *the relation associated with a given collection of subsets* and *the collection of subsets associated with a given relation*. Therefore, I start by giving a reading assignment that goes through Problem 4.3.13 while students are still presenting problems from section 4.2. I spend 15 minutes at the end of a class period discussing Problems 4.3.12 and 4.3.13. In an effort to be as concrete as possible in 4.3.13, we actually poll members of the class to get the information needed for the problem. Then we write down specific subsets and ordered pairs. In closing, I tell them to take another stab at any problems they couldn't do before and to do so as soon as possible (immediately after class is best!). I also assign problems 14 and 15 (Page 99). All of these help them to cement their understanding of the definitions.

By this time, students are fairly comfortable with the transition from relations to subsets and back. They have proved that a relation generated by a collection of subsets. They are ready to show that a collection of subsets whose union is the underlying set yield a reflexive relation and that a pairwise disjoint collection yields a transitive relation. I have them work out proofs for 4.3.15–4.3.17 and present those results in class. These establish finally that the relation induced by a partition is reflexive, symmetric and transitive, thus making the definition of equivalence relation a natural one. Though some students struggle through this, there always seem to be volunteers ready to show their work. After the students have understood 4.3.17, we talk about the significance of Lemma 4.3.20 and Theorem 4.3.21 (an equivalence relation gives rise to a partition), and I assign these to student as a written assignment. I may give them some class time to work on these in small groups, but I let them work on it outside of class for a bit first.[17] In class we discuss the notion of equivalence classes pretty thoroughly. I make sure that Exercise 4.2.23 is well understood, perhaps by alternating discussion with short (2-3 minute) work times in class. If time permits, I have them work through some or all of Problem 16. I always assign Problem 17 (the rational numbers as equivalence classes of pairs of integers) because it lets the students know that in some limited contexts they have

[17] Some students will struggle with this, but I think it is very important for everyone in the class to completely internalize the details of this argument. It makes sure they really understand the notation and the ideas. Both of which are very important.

been working with equivalence classes for years. Furthermore, gives them some insight into the use of the word "equivalent" in this context.

Section 4.4—Graphs

The section on Graphs is self-contained and may be included or not. It is new to the second edition, so I have not yet had a chance to use it in a class. I included it for several reasons. First of all, this is a nice topic that shows a different and very important application of relations. It is a topic that students really like. And it immediately provides a nice variety of accessible (while still non-trivial) results for the students to work on. This is an especially auspicious place to show the usefulness of mathematical induction. It is easy to draw pictures and there is a sufficient variety of interesting behavior in graphs that students can (and do!) make interesting conjectures. Better yet, the proofs (or disproofs) of those conjectures are often within their reach.

Time may not allow me to cover Section 4.4 in the course that I teach, but I would really like to. (It will certainly be the first thing that gets covered if I have some time at the end of the semester!) I may well assign a small piece of it on a takehome exam—this is a good way to test my students' increasing ability to read a definition or two and bring to bear proof techniques and strategies they have been learning. (The difficulty here is the number of words that need to be defined in order to be able to say something of consequence!)

Chapter 5—Functions

The first few sections of Chapter 5 really seem to bring things together for the class. This is partly due to the fact that students are becoming a bit more relaxed about the idea of proving theorems and partly to the fact that they contain many propositions with accessible proofs. Most of my students find that with some work, they can really get these proofs, and there are enough so that many different students can have a turn at the board. The fact in the study of functions students start to see real connections to their previous mathematical experience is also very important. Even the most skeptical students begin to see why someone[18] would actually care about this.

Section 5.1—Basic Ideas

The definitions of function, domain, codomain, range, one-to-one and onto seem to come fairly easily to students. I usually spend one day on this section in which we talk through or present the exercises that students were asked to

[18]For the really hard core applied people, it still may not be them, mind you, but progress is progress!

do when they read the section. Important questions are answered as a matter of course in doing this.

There is one major addition for the second edition. After the first edition of *Chapter Zero* was published, I needed one more "really easy" problem for a takehome exam I was writing. I gave my students an explicitly defined function from $\mathbb{R}$ to $\mathbb{R}$ and asked them to show it was one-to-one and onto. I made sure the algebra involved was straightforward. I was flabbergasted that few students had any idea even how to begin! Despite the fact that we had been very explicit about how one should proceed when trying to show a function was one-to-one or onto, the students were not able to see the connection between our approach to proving that "the composition of two one-to-one functions is one-to-one" and proving the same thing for an explicitly defined function. They needed to see a wider variety of applications to get the general message. As a result, the second edition includes some straightforward problems involving explicitly defined functions. (Exercises 5.1.14 and 5.1.15; Problems 1-3 at the end of the chapter.) These can be skipped, if necessary, but I strongly recommend them. In a related concern about lack of intuition about how these notions apply to real valued functions, I also added Exercises 5.1.4 and 5.1.11. These exercises make students explicitly articulate the connections between intutitive ideas and the mathematical definitions that are being considered.

Sections 5.2 and 5.3—Inverses, Images and Inverse Images

Sections 5.2 and 5.3 ask the students to prove a large number of theorems; fortunately, most are accessible and even students who have been stumbling to this point find they can really sink their teeth into these. They can draw meaningful pictures, and a careful articulation of what they see in the picture usually constitutes a proof, or nearly so. It is a time when the hard (and sometimes frustrating work) of earlier in the semester really pays off. I recommend pointing this out to your students. They appreciate the praise, but more importantly, they surprise themselves by realizing that you are right. Things *have* slipped into place and proving theorems *has* gotten easier. With a little nudge from you they can see just how far they have come in a few short weeks.

The downside of the large number of propositions is that the sections invariably take quite a while to get through. Even if students present one proof right after another, I can count on staying in 5.2 and 5.3 for a while. I have found that if I have all of the students working on all of the assigned proofs, as is my usual custom in other sections, volunteers fight for the first few and then things bog down.

My standard solution to this slowing effect is to divide the class into groups of three or four and divide up the theorems among the groups. This strategy keeps things going pretty well, and has a secondary positive effect. The less well-prepared, intimidated (or lazier) students can no longer sit back and let the best students present all the proofs. Since they have specific proofs that they know they will be responsible for, they end up working on them with perhaps more determination than they have shown to this point. It works very well. (See the discussion starting on Page 26.)

> **A Tip:** There are a large number of problems at the end of the chapter that can be used to supplement the work in this section. Basically, I have put all the good test questions I came up with over the years at the end of the chapter as problems. I am not sure what I will do for test questions, but I think the book is the better for it!

The Rough Spots

Theorems 5.2.7 and 5.2.9 deal with the existence and uniqueness of the inverse function. This was stated much more compactly in the first edition of *Chapter Zero*, but there was so much going in the theorem on that my students found it impenetrable. In the last few years, I have made it a habit to present my own proof of the theorem in a lecture format. This has had several advantages. It showed the students a well-presented proof of a fundamental result and allowed us to move ahead relatively expeditiously. It came at a point in the semester when students no longer considered it "my job" to lecture to them. Thus when I did, they viewed it as a treat and had no problem going back to presenting their own work when it was over. Furthermore, I could assign other proofs to the groups and give them some time to work on their problems before they had to come back with completed proofs. For the second edition, I have broken up the theorem into smaller, "bite-sized" pieces that I hope will be more tractable for the students. I will probably assign various parts of these to the groups. But lecturing on this is still a good option, I think.

Some of my students have had problems with language in Theorems 5.3.6 and 5.3.11 (images and inverse images of intersections and unions)—mainly, this follows from not understanding and using the definitions. For instance, in the problems that deal with inverse images, there is always a large contingent that tries to muddle through by slinging notation around. Invariably (despite the text's warnings to the contrary) this results in an implicit assumption that f^{-1} is a function, and everything falls apart from that point on. Placing some emphasis on Exercises 5.3.3 and 5.3.9 should help with this problem. I believe the issue is mostly a deficit of language which is precisely what these simple exercises deal with.

Problem 5.3.12(2) (when is the inverse image of the intersection the same as the intersection of the inverse images?) can be problematic because students ignore the phrase "for all choices of $\{T_\alpha\}_{\alpha\in\Lambda}$."[19] It is worth pointing out the significance of this statement to the students that are working on the problem.

[19] This is not too surprising. They have done precisely this in propositions 5.3.6 and 5.3.11 and nothing bad has happened. Here it is really an issue for the first time.

(I also mention that they had better use the fact that f is one-to-one when proving that

$$\bigcap_{\alpha \in \Lambda} f(T_\alpha) \subseteq f(\bigcap_{\alpha \in \Lambda} T_\alpha).$$

The careless ones often convince themselves with a false proof that doesn't use the hypothesis.)

Section 5.4—Order Isomorphisms

On the one hand, this is an easy section to skip if the class agenda demands time and attention elsewhere. On the other hand it is a short section, and I like to include it for a couple of reasons.

First of all, the students immediately begin to build on the ideas of the previous two sections. Here they see one-to-one correspondences in action. Second, and more importantly, studying order isomorphims sows a seed that may well prove fruitful in the students' further mathematical studies: the general concept of one-to-one correspondences that preserve mathematical structure. Students often find the concept of isomorphism hard to penetrate. The more different notions of isomorphism that students see, the more they will be able to grasp what is going on. I believe that order isomorphism is a good place to start; the lattice diagrams allow students to clearly see why we call these identical mathematical structures.

Section 5.5—Sequences

Though all of my students have seen sequences in a calculus course, I found them unprepared to handle sequences and sequence notation at the level that I expected in a Real Analysis course. Seeing this, furthermore, as a venue in which students could further expand their notion of the concept of function, I decided to include this section in the book.

After I did, I found a number of places where I could use the notion in later chapters. It plays an important role in the chapters on Cardinality and the Real Number System. This section, however, mostly establishes notation and language. I cover it quickly and reinforce important notions when they come up later on.

Subsequences and subsequence notation, in particular, seem to give my students some difficulty. The essential idea is a very simple one, but the notation required to express this idea precisely the students find to be more complex. Thus the discussion of subsequence notation is expanded to be more explicit in the second edition of *Chapter Zero*. I also gave an expanded discussion of how to use mathematical induction in the construction of subsequences.

Subsequences, however, do not play a large role in the remaining chapters of the book, so (from the point of view of this book) these sections can safely be skipped, if you are willing to let the occasional appearance of subsequences be handled somewhat loosely.

Section 5.6—Binary Operations

This section is best covered quickly. Though the ideas discussed are important ones, they are not deep. In fact, things like commutativity and associativity ought to be known to the students already. What this section brings to students is a mathematical framework for familiar ideas and perhaps a slightly different perspective on some of those ideas.

The exercises should be straightforward (though a few will require a little thought/work), and the students should be able to grasp the basic ideas without too much trouble.

I assign the section to be read, and spend some time answering questions and going over exercises in class.

Chapter 6—Elementary Number Theory

My goals in this chapter are at least as much algebraic as number theoretic. Though I could have chosen to approach Number Theory from a variety of different perspectives, I chose to concentrate on the theme of divisibility. It naturally leads to some important number theoretic concepts, and it also represents a useful first step for students who will eventually end up in an abstract algebra course.[20]

The final discussion of divisibility in $\mathbb{Z}_n$ gives the students the key ingredient of the proof that $\mathbb{Z}_p$ is a field under the operations of addition and multiplication modulo n, and it reveals the essential number theoretic underpinnings of that fact. Moreover, it will give them some insight into distinctions between fields and rings: *Now you can multiply. When can you divide?* Many ideas in elementary abstract algebra have roots in number theory, yet this "ancestry" is often unclear to students. Covering this chapter, will allow them to explore these ideas with, perhaps, a bit more leisure than is to be had in an Abstract Algebra course.

The ideas are developed cumulatively; thus there are not so much natural breaks in the chapter as stages along the way.

- The well-ordering of $\mathbb{N}$.
- The division algorithm.
- Divisibility, prime numbers, divisibility as a partial order.
- Common multiples and common divisors.

[20]The natural jump from integer to modulo arithmetic will give students a leg up on thinking about quotient structures and well-definedness of operations defined on equivalence classes. More importantly, once students who have studied from *Chapter Zero* encounter an abstract quotient structure, I hope they will see it as the obvious generalization of a mathematical structure that crops up (fairly) naturally in the integers.

- The Euclidean Algorithm.
- Relatively prime integers, greatest common divisor as a linear combination.
- The fundamental theorem of arithmetic.
- Congruence modulo n.
- Addition and Subtraction of Congruence classes modulo n. (Including well-definedness of the operations.)
- When do partitions of $\mathbb{Z}$ yield well-defined operations?
- Divisibility modulo n.

Saving Time

The Euclidean algorithm and the fundamental theorem of arithmetic can, in principle, be skipped since they are not subsequently used in the book.

Chapter 7—Cardinality

Sections 7.1 and 7.2—Galileo's paradox and Infinite Sets

Depending on the amount of time I have, my goals in chapter 7 vary. If I am rushed, my goals are very modest and I mostly try to get across the "basics" about countable and uncountable sets. Roughly speaking: the natural numbers and and the rational numbers are countable, the real numbers are not. If time is short, I want to spend most of that time in sections 7.3 and 7.4.

Nevertheless, I do some time in sections 7.1 and 7.2. These set the stage for some of the issues that will be considered later and they help ease the students into the ideas. There is really only one thing in these two sections that will be time consuming, and that is the characterization of infinite sets (Theorem 7.2.5).

Saving Time

Saving Time Despite the generous hints, students find the characterization of infinite sets to be fairly challenging. So when I ask the students to work on the proof, I usually divide the class into four groups and have each group work out one piece of the proof. When I don't want to take a lot of time on Theorem 7.2.5, I adopt one of three time-saving measures.

Fast I write up the proof myself and present it lecture-style.

Faster An intermediate strategy is to elaborate on the hints given for the proof with some pictures and other explanations that help the students see how the arguments would work if they tried to work out the details. When I do this, I may hand out a written proof that takes care of the details.

Fastest I limit myself to discussing the informal scenarios that motivate the theorem and skip the proof.

Sections 7.3 and 7.4—Countable and Uncountable Sets

The information in sections 7.3 and 7.4 is pretty much the standard stuff on cardinality. It is worth noting that the proof of Theorem 7.3.5 requires the Axiom of Choice. I make no mention of this in the book. (Most students will appeal to this intuitive notion without a second thought.) Nevertheless, if time permits, it is nice to bring up an informal discussion of the Axiom of Choice, at this point.

Cantor's diagonalization argument is fully written out in the section. In a book that requires students to devise virtually all the proofs themselves, students aren't forced to develop the (non-trivial) skill of reading a "textbook proof." This gives them one really good chance to do that. In my experience, some students will have trouble understanding the write-up. I try not to explain it to them. As usual, I try to lead them through the argument, perhaps helping them construct an example.

The Rough Spots

The generalized Cantor argument (Theorem 7.4.8) tends to throw students for a loop. Exercise 7.4.7 is new in the second edition. I hope it will help students with the notation. However, even those students who could make it through the logic (I give a pretty big hint, so many of them do), have a heck of a time seeing any connection between Cantor's diagonalization argument and this one. Looking at the form the argument takes when A is $\mathbb{N}$, can really help. It can also help to make Problems 7.4.9 and 7.4.10 more meaningful.

Consider any function f from nat into $\mathcal{P}(\mathbb{N})$. It would "look" something like this:

$$\begin{array}{ccl} 1 & \longrightarrow & \{1,3,4,5,\ldots\} \\ 2 & \longrightarrow & \{1\} \\ 3 & \longrightarrow & \{2,4,6,8,10\ldots\} \\ 4 & \longrightarrow & \{3,4,5,6,7,\ldots\} \\ \vdots & \longrightarrow & \vdots \quad \vdots \end{array}$$

Given Exercise 7.4.7, students should be able to (at least partially) identify the set J that is obtained using the diagonaliztion procedure, but this doesn't really help them to connect this with Cantor's argument.

Suppose we associate each subset of $\mathbb{N}$ with a sequence of 1's and 0's as in Problem 7.4.9. [21]

	1	2	3	4	5	6	7	8	9	10	11	12	...
$\{1\}$	1	0	1	1	1	1	1	1	1	1	1	1	...
$\{1\}$	1	0	0	0	0	0	0	0	0	0	0	0	...
$\{2, 4, 6, 8, 10, \ldots\}$	0	1	0	1	0	1	0	1	0	1	0	1	...
$\{3, 4, 5, 6, 7, \ldots\}$	0	0	1	1	1	1	1	1	1	1	1	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...

Now if we look at the chart corresponding to our function f above, we see that we look at the nth slot on the nth line and "reverse it." That is, if the nth slot on the nth line is a 1, J will have a 0 in the nth slot. If it is a 0, J will have a 1 there. It is now easy to see why this is a generalization of Cantor's diagonalization argument.

Sections 7.5 and 7.6—Comparing Cardinalities

Though well-definedness may require a bit of discussion (especially if you have not previously covered modulo arithmetic in chapter 6), the only part of Section 7.5 that will give the students serious pause is the proof of the Schroeder-Bernstein theorem. Even with the detailed proof sketch, most will have to struggle to see what is going on, and struggle again to write the details. I find it worth my while to spend some time discussing the pictures in the proof sketch so that students will understand what the issues are and how the idea of this proof resolves those issues.

I hope that students will be sufficiently intrigued by the discussion of the continuum hypothesis in Section 7.6 to go off in search of more. Even those students that don't feel compelled to find out more will be made aware of one of the more famous undecidable propositions of set theory and, more generally, of Godel's theorem.

There is one subtle mathematical issue that you may want to bring up with your students while discussing the proof of Theorem 7.6.1 ($\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$ have the same cardinality). Why can't we just construct a one-to-one correspondence between the characteristic functions of the subsets of $\mathbb{N}$ and the binary expansions of numbers in $(0, 1)$? The answer, of course, is that the binary expansions $.1000\ldots$ and $.01111111\ldots$ represent the same real number. However, the sets $\{1\}$ and $\{2, 3, 4, 5, \ldots\}$ are by no means the same.

[21] This exercise can help to make the idea of a characteristic function a bit more concrete for the students, and they can be reminded that a function from $\mathbb{N}$ to $\{0, 1\}$ is just a sequence of 0's and 1's.

Chapter 8—The Real Numbers

Sections 8.1 through 8.4—The Axioms

When I set out to write about the axioms for $\mathbb{R}$, I wanted to do more than just list them and go on to prove theorems. I wanted students to come away with a strong sense that the various axioms are there to accomplish very specific jobs, so I spent a lot of time (indeed most of the time) talking about how we decide what axioms to choose. The words I use in the text are *desirability* and *necessity*. The axioms we choose are desirable because they assign to the real numbers properties that we intuitively know they should possess. How do we know they are really necessary? Before we choose each axiom, we show that the choice is necessary by demonstrating that the previously chosen axioms are insufficient to ascribe familiar properties to the real numbers.[22]

It is quite clear that this chapter will teach the students no new "facts" about the real numbers, though they may come to understand how certain familiar ideas fit together. I think it is important to let the students know that the goal here is not to surprise them with the fact that $a \cdot 0 = 0$ for all $a \in \mathbb{R}$. This may be a good time to revisit the question, why do we bother to prove things that we already "know" to be true? (Students are sometimes too polite or intimidated to ask this question, but it is certain that some of them are thinking it!)

One of the goals of the chapter is clearly to establish for the real numbers a solid axiomatic foundation that will make their study subject to the rigorous mathematical analysis the students have just spent a whole book learning; but pedagogically it goes beyond that. The material in the chapter allows for a general discussion of several interesting issues in abstract mathematics. The book allows students to examine, in a familiar case, what sorts of considerations go into the the construction of a set of axioms. It is an opportunity to talk about differences between an abstract axiomatic system and a concrete object that it is meant to describe.

An underlying principle is this: If two different structures both satisfy a given list of axioms, we will be unable to prove (assuming no more than those axioms) assertions about one of them that do not also hold true for the other. This idea is a fairly subtle one for many students.[23] Understanding this principle is also a

[22]When do we stop? The chapter begs the question of how we know when we are through choosing axioms. The axioms we chose certainly established all the properties of $\mathbb{R}$ that we listed in section 8.2, but one could easily keep making observations about the real numbers until the cows come home. How do we know that by doing this we won't find the need for yet another axiom? The answer is that, without some further high-powered mathematics, we don't. To show that a set of axioms is sufficient to fully describe a given object, we must show that any two structures that satisfy these axioms must be isomorphic (in some sense that is not specified in the book). It requires a complete characterization such as the one given for the rationals (as a totally ordered set) in Question to Ponder 4 on page 177.

[23]Some students may fall into the trap of thinking that if you can't prove a given statement, then it must follow that you can prove its negation. Watch for this and be prepared to make the distinction between false and unprovable statments.

very important step toward understanding some of the reasons for mathematical abstraction. The study of a given axiomatic system can teach us some important lessons. For one thing, it enables us to sort out properties that are shared by otherwise very different mathematical structures. More to the point, limiting our assumptions clearly exposes chains of mathematical dependence. The chain of reasoning in chapter 8 shows that the field axioms and the order axioms alone do not allow us to prove that all positive real numbers have a square root. This is not merely due to a lack of sufficient cleverness. We have shown that it cannot, in principle, be done.

Section 8.5—Sequence Convergence

The final section of chapter 8 is there to allow the students to use the structures that they have built up throughout the book. Pedagogically, it is also a sort of "pre-analysis" section. The students are briefly exposed to some "epsilonics," and the discussion surrounding the definition of a metric space sets the stage for some of the standard abstractions that students will encounter in an elementary analysis or topology course.

The Appendices

Much is made of **THE AXIOMS OF SET THEORY** both in *Chapter Zero* and in other mathematics books. However, elementary treatments of the building of the real numbers from set theory are rare. Books on axiomatic set theory may do the job, but they also spend a lot of time on broader issues that may obscure (for undergraduates) the main point: a handful of fairly straightforward axioms about sets allow us to build all the familiar mathematical characters. Naturally, we use our intuitions about how those characters ought to "look" and "behave" in the construction, but it can be fully carried out without assuming anything beyond the few simple set axioms.

Students who attempt to delve into this section will find that the real numbers are genuine mathematical monsters. The tower of reasoning that is required to build them is amazingly complex—it is a good thing that we simplify notation as we go along. (Have you ever tried to write down the real number "2" explicitly as a set?!)

The two appendices are written in a style similar to the rest of the book. That is, the students are led through the ideas, but the proofs are left for them to fill in. However, the appendices are considerably sketchier than the earlier parts of the book. This is particularly true in appendix B, where the proofs of many of the properties of the various constructions can be quite involved. Furthermore, the appendices often rely on the development in the main text. (For example, once the Cartesian product of two sets is constructed using the axioms, the entire development of relations and functions in chapters 4 and 5 may be assumed.)

Nevertheless, the development in appendices A and B should be clear enough to allow a bright student to understand the broad outlines, if not all of the technical details.

Part III
Technical Details

Elaborations on the Dependency Chart

Skipping Partial Orderings

Section 4.2 looks at the first important special type of relation—the partially ordered set. Some further parts of the book rely heavily on Section 4.2, most especially the discussions of order isomorphism (5.4) number theory (chapter 6), comparing cardinalities (7.5 and 7.6), and the order and least upper bound axioms for $\mathbb{R}$ (8.3 and 8.4). Though the sections on sequences nominally depend on Section 4.2, it is easy to cover these without 4.2 by telling students to replace any references to totally ordered sets by $\mathbb{R}$ under the usual ordering.

If you don't plan to cover order isomorphisms, number theory, or the axioms for the real number system, and you plan to do at most the "basics" of cardinality, you can skip this section altogether.

Skipping Sequences

Sequences play a sizable role in the chapter the real number system. They play a central role in the chapter on cardinality. However, if you do not plan to include a study of cardinality, you can avoid sequences altogether. It would be possible to cover some of the ideas of cardinality by handling the ideas about sequences at the intuitive level. The chapter on number theory and the first four sections of chapter 8 on the Real number system don't depend on sequences. (Sequences don't show up in chapter 8 until section 8.5—after all the axioms have been introduced.)

Cardinality—a Small Caution

The dependency chart shows that Section 7.5 depends on the material in section 4.3 (equivalence relations), but that section 7.1 does not. This is not strictly true; 7.1 does depend very tenuously on equivalence relations. Exercise 7.1.3 asks the students to show that, for any set X, "has the same cardinality as" is an equivalence relation on $\mathcal{P}(X)$—though the various pieces of this are separated out into separate propositions for the second edition. The final statement of the theorem concludes that the relation "has the same cardinality as" is an equivalence relation.

The ideas in section 7.5, however, depend in a fundamental way on an understanding of equivalence relations. Thus I decided to emphasize this more important connection by drawing the arrow from 4.3 to 7.5 instead of 7.1.

The upshot of this is that if you wish to cover the basics of cardinality without covering equivalence relations, you can easily do this by telling the students to ignore the concluding statement of Theorem 7.1.3. You will not, however, be able to progress into a general discussion of comparing cardinalities.

Errata

Here are the currently known errors in *Chapter Zero, 2e* as of 2/5/2002. As more errors are discovered, I will post updated versions of the errata list at

http: \\ math.kenyon.edu \ schumacherc \ Professional \ Research \ Errata.pdf

If you find errors, I would appreciate hearing about them.

- Pg. xii (Preface)—the last two words of the third line should be "second edition" instead of "first edition."

- Pg. 7—Beginning of the next to the last paragraph should read "Your goal is *to* figure out ...." The word "to" was omitted.

- Pg. 30—Example 1.10.1 ... "Assume that $x^3 + 37$ has a real root ..." should read "Assume that $x^3 - 37$ has a real root." Alternatively, all the minus signs in the proof should be plus signs.

- Pg. 37—Problem 5(c). This was a bad error on my part. The problem, as it is written, is very misleading, if not exactly wrong.

  The quick fix: change the $\vee$ in the second expression to a $\wedge$. Unfortunately, this defeats the purpose of the problem. It does not forcefully put across the message that if students are trying to prove a statement of the form "A implies 'B or C' " they will not be able to consider the statements "A implies B" and "A implies C" separately.

The long story: As you no doubt know, the statement "A implies 'B or C' " is not generally equivalent to the statement "A implies B, or A implies C." Consider, for instance, the statements

- If x is an integer, then x is even or odd.
- If x is an integer, then x is even.
- If x is an integer, then x is odd.

The first is true, but neither the second nor the third is.

Nevertheless, a naive use of truth tables will (apparently) show that the statements

$$(\mathrm{A} \implies (\mathrm{B} \vee \mathrm{C})) \text{ and } (\mathrm{A} \implies (\mathrm{B}) \vee (\mathrm{A} \implies (\mathrm{C})$$

are equivalent. So how do we deal with this?

The key is in the assumed quantifier that goes along with an implication. As it happens, the statement

$$\text{For all } x(\mathrm{A}(x) \implies (\mathrm{B}(x) \vee \mathrm{C}(x)))$$

is equivalent to

$$\text{For all } x((\mathrm{A}(x) \implies \mathrm{B}(x)) \vee (\mathrm{A}(x) \implies \mathrm{C}(x))).$$

It is *not* equivalent to

$$\text{For all } x(\mathrm{A}(x) \implies \mathrm{B}(x)), \text{ or for all } x(\mathrm{A}(x) \implies \mathrm{C}(x)).$$

It is this latter interpretation that we place on the statement "Either A implies B, or A implies C."

A truth table can be used to show this, as follows: ask students to fill in the following truth table:

A	B	C	$\mathrm{A} \implies \mathrm{B}$	$\mathrm{A} \implies \mathrm{C}$	$\mathrm{A} \implies (\mathrm{B} \vee \mathrm{C})$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Have the students note that the only line of the table for which $\mathrm{A} \implies (\mathrm{B} \vee \mathrm{C})$ is false is the third line. If all values of the variable keep us away from the third line of the table, we are guaranteed to be in a situation where if $\mathrm{A} \implies \mathrm{B}$ is false then $\mathrm{A} \implies \mathrm{C}$ is true, or vice versa. However, staying away from the third line does not guarantee the truth of $\mathrm{A} \implies \mathrm{B}$, nor does it guarantee the truth of $\mathrm{A} \implies \mathrm{C}$.

- Pg. 54—Problem 4, parts (b) and (c). The range of the indexing sets should be the **positive** rational numbers.
- Pg. 58—In the middle of the page in the second line of the paragraph following the "theorem,"— "For $n = 41, \ldots$" should read "For $n = 40, \ldots$.
- Pg. 97—Problem 4. The last sentence should read: "Which of these relations do economists assume to be anti-symmetric? Explain."
- Pg. 111—Theorem 5.2.7. The first sentence defines a function from the set A to the set B. This function should, instead, go from B to A. In other words, the first sentence should read "Let $f : B \to A$ be a function."
- Pg. 114—Exercise 5.3.9. There is an extra comma in the offset text. It should read

  $c \in f(Y)$ provided that ...
- Pg. 211—The third line in the grey box is missing a letter. The word "dfficult" should instead be "difficult."
- Pg. 133 — Problem 16. The second sentence reads:

  Define $\mathcal{F} : \mathcal{P}(B) \to \mathcal{P}(B) \ldots$

  It should, instead, read:

  Define $\mathcal{F} : \mathcal{P}(B) \to \mathcal{P}(A) \ldots$