

Number Theory

Chapter Perspective

Number theory is what we discover and verify about number properties, number patterns, and number relationships. The fascinating ideas of number theory have many practical and technological applications. For example, number theory is used in cryptology, the science of devising and decoding coded messages. The cryptology machine in the above photo was developed by the U.S. military many years ago to help in these processes. Today, much more sophisticated cryptology technology is available. Number theory, in addition to being an inherently interesting field of study, also forms the basis for the study of other important mathematical topics.

In this chapter, you can develop a greater sense of numbers and their relationships by exploring ideas associated with factors, multiples, divisibility, prime and composite numbers, and the greatest common factor and least common multiple properties.



Connection to the NCTM Principles and Standards

The NCTM *Principles and Standards for School Mathematics* (2000) indicate that the elementary school mathematics curriculum grades PreK–8 should include the study of number theory so that students can

- develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers (p. 78);
- describe classes of numbers according to their characteristics such as the nature of their factors (p. 148); and
- use factors, multiples, prime factorization, and relatively prime numbers to solve problems (p. 214).

Connection to the PreK–8 Classroom

In grades PreK–2, students build number sense and readiness for number theory ideas. Expressing 6 as 2×3 helps them build the concept of factors. Skip counting by 5's helps them build the concept of multiples, and so on.

In grades 3–5 students build rectangular arrays with chips to investigate odd, even, and prime numbers, and use the idea of factors to classify numbers.

In grades 6–8 students utilize number theory ideas, such as prime numbers and divisibility properties, to discover patterns, reason about relationships, and solve problems.

Section 4.1**FACTORS AND DIVISIBILITY**

- Connecting Factors and Multiples
- Defining Divisibility
- Techniques for Determining Divisibility
- Using Factors to Classify Natural Numbers

In this section we examine the idea of a factor of a number from several viewpoints. We also look at techniques for finding the factors of a number and use these techniques to develop shortcuts for deciding whether a number has a certain number as a factor. Finally, we classify natural numbers according to the number of factors they have and analyze these classifications.

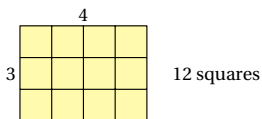
Connecting Factors and Multiples

In general number theory is the study of the characteristics of and relationships involving the natural numbers. Many of the useful characterizations of the natural numbers are based on information about the *factors* and *multiples* of a number.

The labels on the following equations highlight the ideas of factors and multiples.

$$\begin{array}{ccc}
 \text{Factors} & & \text{A multiple of 3} \\
 \text{of 12} & & \text{A multiple of 4} \\
 \swarrow \quad \searrow & & \downarrow \\
 3 \times 4 & = & 12.
 \end{array}$$

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The yellow rectangular array models the idea of factors and multiples described above. It shows visually that 3 and 4 are factors of 12, and that 12 is a multiple of 3 and 4.

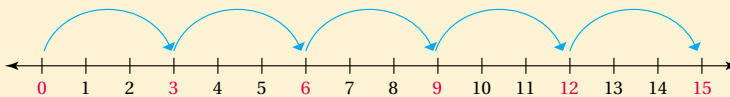
Moving from the above examples to a general statement, we give the following definition.

Definition of Factor and Multiple

If a and b are whole numbers and $ab = c$, then a is a **factor** of c , b is a factor of c , and c is a **multiple** of both a and b .



The NCTM *Principles and Standards for School Mathematics* state that “Teachers should help students (primary grades) represent aspects of situations in mathematical terms, possibly using more than one representation” (p. 141). Representing skip counting as “jumps” on the number line can help children begin to understand the ideas of a factor and a multiple of a number. For example, each jump of three units on the number line below lands on a *multiple* of 3. The number line also shows vividly that 3 is a *factor* of 15, because $3 \times 5 = 15$.



Finding Factors and Multiples. You can mentally produce *multiples* of small numbers. For example, counting by fives, 5, 10, 15, 20, ... produces multiples of 5. Calculator displays also provide a way to multiply a factor to produce a list of multiples of the factor. Comparable keystrokes and displays from a four-function calculator and a graphing calculator that show multiples of 3 are as follows.

■ A four-function calculator

Key Sequence	Display
$\boxed{+} \ 3 \ \boxed{=}$	3
$\boxed{=}$	6
$\boxed{=}$	9
$\boxed{=}$	12
$\boxed{=}$	15

■ A graphing calculator

Key Sequence	Display
0 $\boxed{+} \ 3 \ \boxed{\text{ENTER}}$	0+3 3
$\boxed{+} \ 3 \ \boxed{\text{ENTER}}$	ANS+3 6
$\boxed{\text{ENTER}}$	9
$\boxed{\text{ENTER}}$	12
$\boxed{\text{ENTER}}$	15

Using these calculators, we can extend the list of multiples as much as we want or easily produce a multiple of any number.

You can find the *factors* of a number by dividing. A simple way to decide whether a first number is a factor of a second number is to divide the second number by the first. If the quotient is a natural number with a remainder of 0, the divisor and quotient are each factors of the dividend.

With larger numbers, a calculator can help identify whether one number is a factor of another. Some calculators have what is called an *integer division feature*, $\boxed{\text{INT} \div}$, which gives the quotient and remainder for a division. We use the following keystrokes and display from a calculator that uses integer division to determine whether 24 is a factor of 5868.

Key Sequence	Display
5868 $\boxed{\text{INT} \div}$ 24	24
$\boxed{=}$	244 12
	└─ Q ─┘ └─ R ─┘

Because the remainder is 12, not 0, 24 isn't a factor of 5868. Example 4.1 illustrates how making a decision about a factor of a number can be useful in solving a problem.

Example 4.1

Applying the Idea of a Factor of a Number

A movie rental store sold used movie video cassettes for \$13 each, including tax. A clerk recorded a total income of \$1009 for all the cassettes. If the total is possible, how would we know? If it isn't, what would be a possible total?

Solution

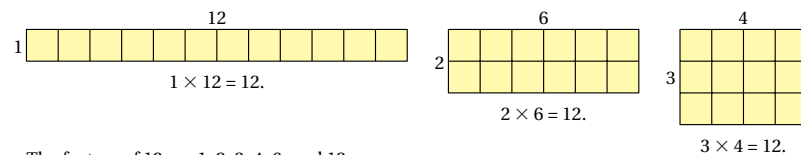
Using a calculator with an $\boxed{\text{INT} \div}$ function, we get the quotient 77 and remainder 8 when we divide 1009 by 13. The remainder isn't 0, so 13 isn't a factor of 1009, and the clerk's total is impossible. A possible total would be \$1001.

Your Turn

Practice: Could computer systems that sell for \$4649 each produce a total income of \$1,111,117? Use a calculator with an $\boxed{\text{INT} \div}$ feature, if possible, to decide.

Reflect: How would you solve the practice problem if a calculator with an $\boxed{\text{INT} \div}$ feature is not available? ■

Finding All the Factors of a Number. Finding all the factors of a number is often useful. Rectangles cut from graph paper may be used to show all the factors of a number geometrically, as illustrated in Figure 4.1.



The factors of 12 are 1, 2, 3, 4, 6, and 12.

FIGURE 4.1 | Geometric interpretation of the factors of 12.

To find all the factors of a small number such as 12, you can divide 12 by each natural number in succession, beginning with 1. Dividing by 1, 2, and 3, you get the factor pairs (1, 12), (2,6), and (3, 4). Continued dividing will produce no new factors, so you find that the factors of 12 are 1, 2, 3, 4, 6, and 12. Using a calculator is a much more effective way of finding the factors of a larger number.

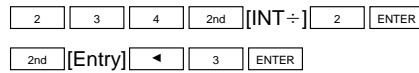
One way to use a scientific calculator to determine the factors of a number is to use its memory key. To do so, first enter \boxed{C} , then enter the number, and then push $\boxed{M+}$ to record it in the calculator's memory. Then try each divisor 2 (as shown), 3, 4, 5, and so on, in order.



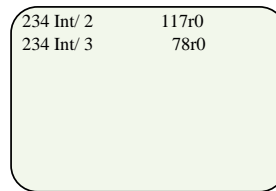
When the answer is a natural number, list the trial divisor and the natural number as factors of the number you put into the memory.

You may also use the $\boxed{INT \div}$ feature to find all the factors of a number. The following keystrokes and display for a graphing calculator show how to begin to find all the factors of 234.

Key Sequence



Display



Repeat the command, checking divisors 4, 5, 6, and so on. If the value for the remainder, R, is 0, the divisor is a factor of 234. To find all the factors of 234, you could use a calculator to divide 234 by successive whole numbers and list the factors you find, as shown, along with 1 and 234.

Number	Divide by	Quotient	Factors
234	2	117	2, 117
234	3	78	3, 78
234	6	39	6, 39
234	13	18	13, 18
234	18	13	18, 13
234	39	6	39, 6
234	78	3	78, 3
234	117	2	117, 2

Thus the factors of 234 are 1, 2, 3, 6, 13, 18, 39, 78, 117, and 234. After many divisions you will have found all the factors of 234, but could you have stopped dividing earlier and had the same information? If so, *how do you know when to stop?* To decide, first note that the numbers in the *divide by* column are increasing and that the numbers in the quotient column are decreasing. The numbers “pass each other” between

13 and 18. A factor of 234 will appear in the divide by column first and reappear later in the quotient column, repeating earlier factors. As soon as the pass-by point is reached, you need check no further because no new factors will appear. When you come to a number already in your factor list—in this case 18—you can stop testing factors.

The pass-by point appears to be the place at which the factors are closest to being equal. Note that, as $\sqrt{234} \times \sqrt{234} = 234$, the pass-by point can't be greater than $\sqrt{234}$. This result suggests the following theorem.

Theorem: Factor Test

To find all the factors of a number n , test only those natural numbers that are no greater than the square root of the number, \sqrt{n} .

Example 4.2 provides an opportunity for you to apply the Factor Test Theorem.

Example 4.2

Applying the Factor Test Theorem

Find all the factors of 165.

Solution

Alison's thinking: I estimated $\sqrt{165}$ by thinking that it was between $\sqrt{144} = 12$ and $\sqrt{169} = 13$, so I only have to test the numbers through 12. I used my calculator and made a list.

- | | |
|-----------------------|-----------------|
| Test 2: No | Test 8: No |
| Test 3: Factors 3, 55 | Test 9: No |
| Test 4: No | Test 10: No |
| Test 5: Factors 5, 33 | Test 11: 11, 15 |
| Test 6: No | Test 12: No |
| Test 7: No | Test 13: No |

The factors of 165 are 1, 3, 5, 11, 15, 33, 55, and 165.

Lee's thinking: I found the square root of 165 on my calculator. It's about 12.85, so I only have to test numbers through 12. I found that 2 wasn't a factor, so I knew that no even number could be a factor. I divided 165 by 1, 3, 5, 7, 9, and 11 and found the factors of 165: 1, 3, 5, 11, 15, 33, 55, and 165.

Your Turn

Practice: Find all the factors of 297.

Reflect: Did Alison have to test the number 13? Explain. ■

Defining Divisibility

Sometimes, as when deciding whether 144 people can be grouped by 3s, we need to make a quick check to see if one number is a factor of another. For example, we ask the question, "Is 3 a factor of 144?" The same question could be asked in other ways, such as, "Is 3 a *divisor* of 144?" or "Does 3 *divide* 144?" or "Is 144 *divisible* by 3?" The idea of divisibility is defined as follows.

Definition of Divisibility

For whole numbers a and b , $a \neq 0$, a **divides** b , written $a \mid b$, if and only if there is a whole number x so that $ax = b$. Also, a is a **divisor** of b or b is **divisible** by a . Further, $a \nmid b$ means that a does not divide b .

We illustrate the definition for 6 and 24 by noting that since there exists a whole number 4 such that $6 \times 4 = 24$, we can say that 6 is a factor of 24, 6 is a divisor of 24, and 24 is divisible by 6.

Techniques for Determining Divisibility

We naturally look for ways to discover and verify conclusions about whether one number is divisible by another. We first present a theorem on divisibility that is often useful in verifying what we have discovered. Consider the following examples. Let's use the facts that $7 \mid 35$ and $7 \mid 14$. Note also that $7 \mid (35 + 14)$, or 49. Similarly, $9 \mid 27$, $9 \mid 36$, and $9 \mid (27 + 36)$, or 63. These examples suggest the following divisibility theorem.

Theorem: Divisibility of Sums

For natural numbers a , b , and c , if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

We can verify this theorem as follows. Suppose $a \mid b$ and $a \mid c$. Then from the definition of divisibility, we know that there exist whole numbers r and s such that $b = ar$ and $c = as$. By adding these equations we see that $b + c = ar + as$. It follows that $b + c = a(r + s)$, and since $r + s$ is a whole number, by the definition of divisibility, $a \mid (b + c)$.

Now let's try to discover a theorem about testing divisibility and see if we can use the Theorem on Divisibility of Sums to verify it. Table 4.1 shows how we might use patterns to discover divisibility tests for 2, 5, and 10.

TABLE 4.1 | Patterns for Discovering Divisibility Tests for 2, 5, and 10

Number divisible by 2	Last digit	Number divisible by 5	Last digit	Number divisible by 10	Last digit
0	0	0	0	0	0
2	2	5	5	10	0
4	4	10	0	20	0
6	6	15	5	30	0
8	8	20	0	40	0
10	0	25	5	50	0
12	2	30	0	60	0
14	4	35	5	70	0
16	6	40	0	80	0
18	8	45	5	⋮	⋮
⋮	⋮	⋮	⋮		

Three tests that are suggested by the patterns in Table 4.1 are stated as follows.

Theorem: Divisibility Tests for 2, 5, and 10

- A natural number n is divisible by 2 if and only if its units digit is 0, 2, 4, 6, or 8.
- A natural number n is divisible by 5 if and only if its units digit is 0 or 5.
- A natural number n is divisible by 10 if and only if its units digit is 0.

We verify the statement in the above theorem about divisibility by 2. The statements about divisibility by 5 and 10 can be verified in a similar manner. Let's first look at an example. Consider a number like 274. It can be written as $10(27) + 4$. We see that $2 \mid 10(27)$ and since we know that $2 \mid 4$, by the Theorem on Divisibility of Sums, we know that $2 \mid [10(27) + 4]$, or 274. So it follows that any number is divisible by 2 if its units digit is divisible by 2.

In general, we can express any number n in the form $n = 10q + r$, where q is the quotient when the number is divided by 10 and r is the remainder. Using the Theorem on Divisibility of Sums, we know that if $2 \mid 10q$ and $2 \mid r$, then $2 \mid (10q + r)$, or n . Since we know that $2 \mid 10q$, when we check and find that $2 \mid r$ (the units digit), we know that $2 \mid n$. The only units digits that 2 divides are 0, 2, 4, 6, and 8, so the first statement in the theorem is verified.

The idea of divisibility by 2 is used in the following definition.

Definition of Even and Odd Numbers

- A whole number is **even** if and only if it is divisible by 2.
- A whole number is **odd** if and only if it is not divisible by 2.

Now let's look at the patterns in Table 4.2 to discover another divisibility test, and see if we can use the Theorem on Divisibility of Sums to verify it.

TABLE 4.2 | Patterns for Discovering Divisibility Tests for 3 and 9

Number divisible by 3	Sum of the digits	Number divisible by 9	Sum of the digits
0	0	72	9
3	3	873	18
6	6	8883	27
9	9	77,886	36
12	3	99,999	45
15	6	⋮	⋮
18	9		
21	3		
24	6		
27	9		
⋮	⋮		

Two tests that are suggested by the patterns in Table 4.2 are stated as follows.

Theorem: Divisibility Tests for 3 and 9

- A natural number n is divisible by 3 if and only if the sum of its digits is divisible by 3.
- A natural number n is divisible by 9 if and only if the sum of its digits is divisible by 9.

We can use the Theorem on Divisibility of Sums to verify that the divisibility test stated above for 3 works. The divisibility test for 9 can be verified in a similar manner. Consider any 3-digit number, expressed in the form $a \times 100 + b \times 10 + c$. It can also be written as $a(99 + 1) + b(9 + 1) + c$, or as $(99a + 9b) + (a + b + c)$ we know that $99a + 9b$ is divisible by 3, and by the Theorem on Divisibility of Sums we can conclude that if $a + b + c$ is also divisible by 3, then the original 3-digit number is divisible by 3. And this last conclusion is the test for divisibility by 3, stated above.

We use some of the divisibility tests and the problem-solving strategy *use logical reasoning* in Example 4.3.

Example 4.3

Using Divisibility Tests

A manufacturer wants to put 3 items into each package and then 2 packages in each box. The packing supervisor wants to know whether the 67,986 items in stock will fill a whole number of boxes. Check the divisibility of 67,986 by 3 and by 2 to find out.

■ **Solution**

- The sum of the digits of 67,986 is $6 + 7 + 9 + 8 + 6 = 36$; 36 is divisible by 3, so 67,986 is divisible by 3.
- Because the last digit of 67,986 is the even number 6, 67,986 is divisible by 2.

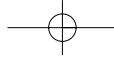
Therefore 67,986 items will fill a whole number of boxes.

■ **Your Turn**

Practice: Suppose that the manufacturer puts 3 items in each package and then 5 packages in each box. Which of the following number of items will fill a whole number of boxes?

- a. 1275 b. 1461
- c. 2770 d. 2535

Reflect: Do you think that the 67,986 items in the example problem would fill exactly 6 boxes? Explain. ■



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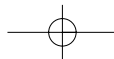
A second important theorem that is useful in verifying certain divisibility tests is suggested by the following. We know that $2 \mid 20$ and that $5 \mid 20$. We can conclude that 2×5 (or 10) $\mid 20$. We state this theorem as follows.

Theorem: Divisibility by Products

For natural numbers a , b , and c , if $a \mid c$ and $b \mid c$, and a and b have no common factors except 1, then $ab \mid c$.

We verify this theorem as follows. If $a \mid c$ and $b \mid c$, we know from the definition of divisibility that $ar = c$ and $bs = c$, where r and s are whole numbers. But since a and b share no factors other than 1, we know from the equations $ar = c$ and $bs = c$ that c contains all of a 's factors and all of b 's factors. Thus c is divisible by ab .

Now let's look at a divisibility test that can be verified using the above theorem. Numbers divisible by both 2 and 3, such as 6, 12, 18, 24, 30, 36, and so on, are also divisible by 6. And any multiple of 6 is also divisible by both 2 and 3. This relationship suggests the following divisibility test for 6.



To verify the “only if” part of this theorem, we note that if a number n is divisible by 6, we can write $6k = n$, where k is a whole number. This can also be written as $2 \times 3 \times k = n$ or $2(3k) = n$, or $3(2k) = n$. Then by the definition of divisibility, we know that n is divisible by both 2 and 3.

To verify the “if” part of the theorem, we note that if $2 \mid n$ and $3 \mid n$, then using the Theorem on Divisibility by Products we can conclude that $2(3)$ (or 6) $\mid n$.

A similar approach could be used to verify the conclusion that a natural number is divisible by 10 if and only if it is divisible by both 2 and 5. We leave this verification for Exercise 58 on p. 204.

The patterns in Table 4.3 are helpful in discovering divisibility tests for 4 and 8.

TABLE 4.3 | Patterns for Discovering Divisibility Tests for 4 and 8

Number divisible by 4	Last two digits	Number divisible by 8	Last three digits
12	12	4160	160
124	24	7248	248
528	28	33,080	080
916	16	59,336	336
1732	32	97,328	328
9044	44	135,864	864
27,356	56	999,832	832
777,508	08		

Note that numbers formed by the last two digits of numbers divisible by 4 are also divisible by 4. Similarly, numbers formed by the last three digits of numbers divisible by 8 are also divisible by 8. These patterns suggest the “if” parts of the following tests.

Theorem: Divisibility Tests for 4 and 8

- A natural number n is divisible by 4 if and only if the number represented by its last two digits is divisible by 4.
- A natural number n is divisible by 8 if and only if the number represented by its last three digits is divisible by 8.

Exercises 59 and 60, pp. 204–205, ask questions to help you verify the above theorem.

The divisibility test for 4 involves mentally dividing a two-digit number by 4, and is fairly easy to use. For example, to decide if 5280 is divisible by 4, we check to see if the number represented by the last two digits of 5280, namely 80, is divisible by 4. Since $80 \div 4 = 20$, Remainder 0, we know that 5280 is divisible by 4.

The test for divisibility by 8 is somewhat more difficult to use, but usually we can mentally divide to use the test. If not, a calculator can be used to make the test slightly easier. For example, to decide if 467,863 is divisible by 8, we check to see if the number represented by the last three digits, namely 863, is divisible by 8. We can mentally divide to see that it is not, and that 467,863 is not divisible by 8. Alternatively, we could use a calculator to find that $863 \div 8$ is not a whole number.

The divisibility tests for 7 and 11 are interesting, but decisions about divisibility for these numbers are more efficiently made with the aid of a calculator. However, to satisfy any curiosity about whether tests exist for all the numbers 1–11, we state and discuss the tests for 7 and 11.

Theorem: Divisibility Tests for 7 and 11

- A natural number n is divisible by 7 if and only if the number formed by subtracting twice the last digit from the number formed by all digits but the last is divisible by 7.
- A natural number n is divisible by 11 if and only if the sum of the digits in the even-powered places minus the sum of the digits in the odd-powered places is divisible by 11.

For example, to test whether 3045 is divisible by 7, we check to see if $7 \mid [304 - 2(5)]$, or $7 \mid 294$. It does, so we conclude from the test that $7 \mid 3045$. Also, to test whether 1,234,607 is divisible by 11, we check to determine whether 11 divides the difference $(7 + 6 + 3 + 1) - (0 + 4 + 2)$, or 11. It does, so we conclude from the test that $11 \mid 1,234,607$. Example 4.4 further demonstrates the use of divisibility tests and the problem-solving strategy *write an equation*.

Example 4.4

Problem Solving: Boxing Baseballs

A sports shop sold baseballs to teams in boxes of 6 and in boxes of 12. The inventory manager reported that the shop had sold 9052 baseballs to teams during the past month. The owner asked the manager to check the inventory, saying that it was incorrect. How did he know?

■ Solution

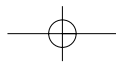
If d is the number of boxes of 12 baseballs and b is the number of boxes of 6 baseballs, then according to the manager's report, $12d + 6b = 9052$. But both $12d$ and $6b$ are divisible by 6, so 9052 must be divisible by 6. The divisibility test for 6 shows that 6 doesn't divide 9052.

■ Your Turn

Practice: Suppose that the baseballs in the example problem came in boxes of 4 and boxes of 8. Could the shop have sold 9052 baseballs? Explain.

Reflect: Which theorem justified the shop owner's conclusion in the example problem that 9052 must be divisible by 6 for the inventory to be correct? ■

Other observations about divisibility suggest some additional useful divisibility theorems. For example, we notice that $6 \mid 36$ and $6 \mid 24$. It follows that $6 \mid (36 - 24)$ also. This suggests the first theorem in the following box, which is similar to the Divisibility of Sums theorem. Also, we note that $4 \mid 12$ and that $4 \mid 3(12)$. This suggests the second theorem in the following box. We also note that $3 \mid (27)$ or $(12 + 15)$ and that $3 \mid 12$. We conclude that $3 \mid 15$. This suggests the third theorem in the following box. The fourth theorem is similar to the third.

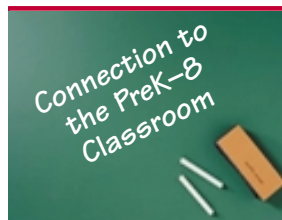


Theorems: Divisibility

For natural numbers a , b , and c ,

- If $a \mid b$ and $a \mid c$, then $a \mid (b - c)$.
- If $a \mid b$ and c is any natural number, then $a \mid bc$.
- If $a \mid (b + c)$, and $a \mid b$, then $a \mid c$.
- If $a \mid (b - c)$, and $a \mid b$, then $a \mid c$.

You will be asked to verify and use one or more of the above theorems in the Problems and Exercises.



The *Principles and Standards for School Mathematics* affirm the role divisibility ideas can play in providing a setting for helping to develop children's reasoning abilities, in statements such as this: "Students can learn about reasoning through classroom discussion of claims that other students make. The statement, 'If a number is divisible by 6 and by 4, then it is divisible by 24' could be examined in various ways. Middle-grades students could find a counterexample—the number 12 is divisible by 6 and by 4 but not 24" (p. 58).

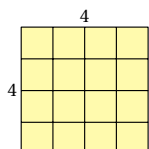
Using Factors to Classify Natural Numbers

Mini-Investigation 4.1 asks you to classify natural numbers according to the number and type of their factors. Such classifications play an important role in the development of some number theory ideas.

Make a table showing your data and another showing your classifications.

Mini-Investigation 4.1 Finding a Pattern

In what ways can you classify the numbers 1 through 30 by considering the factors of the numbers? (*Hint:* One classification might be all numbers that have 2 as a factor.)

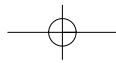


$4 \times 4 = 16.$

FIGURE 4.2 | Geometric model of the square number 16.

One possible classification is that of numbers that have 2 as a factor, commonly called **even numbers**, or numbers that do not have 2 as a factor, called **odd numbers**. Another classification could involve the number of factors a number has. For example, the number 5 has exactly two factors, 1 and 5. Numbers with exactly two factors are important, and we consider them in the next section.

The class of numbers with an odd number of factors could be described as the **squares**. Any square number, such as 16, can be shown geometrically as a square (Figure 4.2). Consider the following classification. A number is called a **perfect number** when the sum of the factors of the number that are less than the number—its *proper factors*—equals the number. The sum of the proper factors of 6, $1 + 2 + 3$, equals 6, so 6 is a perfect number. When the sum of the proper factors of a number is less than the number, it is called a **deficient number**. When the sum of the proper factors of a number is greater than the number, it is called an **abundant**



number. Two numbers are **amicable** if the sum of the proper factors of the first number equals the second number and if the sum of the proper factors of the second number equals the first number. Exercises 30–34, 48, and 49 at the end of this section give you the chance to answer questions about these classes of numbers.

The solution to the problem in Example 4.5 involves the use of a classification of natural numbers just described and the problem-solving strategies *solve a simpler problem*, *make a table*, and *look for a pattern*.

Example 4.5

Problem Solving: The Prize Winners

Prize winners for a giveaway program are to be determined as follows. Envelopes for the first 500 entries will be numbered 1 through 500 and laid out on a long table. The first official will walk along the table, opening all the envelopes. The second official will follow the first and close every even-numbered envelope. The third official will follow the second, reversing every third envelope by closing open envelopes and opening closed envelopes. The fourth official will reverse every fourth envelope, and so on until 500 officials have walked along the table and reversed envelopes. The envelopes that remain open after this procedure will be those of the contest winners! What numbers were on the winning envelopes?

Working toward a Solution

Understand the problem	<i>What does the situation involve?</i>	Choosing prize winners.
	<i>What has to be determined?</i>	The winning envelope numbers.
	<i>What are the key data and conditions?</i>	There are 500 envelopes. The first official opened all the envelopes, the second changed every second envelope, the third changed every third envelope, and so on, through the 500th official.
	<i>What are some assumptions?</i>	The officials walked along the table from the first envelope to the last envelope, following one another in numerical order.
Develop a plan	<i>What strategies might be useful?</i>	Solve a simpler problem, make a table, and look for a pattern.
	<i>Are there any subproblems?</i>	Determine which envelopes are open after the first, second, third, and so on, officials have walked along the table.
	<i>Should the answer be estimated or calculated?</i>	No calculation is needed.
	<i>What method of calculation should be used?</i>	Not applicable.

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Implement the plan

How should the strategies be used?

Suppose that there were only 10 envelopes. Make a table and record the officials' actions.

		Envelope Number									
		1	2	3	4	5	6	7	8	9	10
Official Number	1	O	O	O	O	O	O	O	O	O	O
	2		C		C		C		C		C
	3			C			O				C
	4				O				O		
	5					C					O
	6						C				
	7							C			
	8								C		
	9									O	
	10										C

What is the answer?

Look for patterns in the table that will show which envelopes end up open. Extend the table to 20 envelopes, if necessary, to help solve the problem.

Look back

Is the interpretation correct?

Check the table for completeness and accuracy.

Is the calculation correct?

Not applicable.

Is the answer reasonable?

Look at the data and test the pattern to decide whether the answer is reasonable.

Is there another way to solve the problem?

Logical reasoning could be used to solve the problem by looking at the number of factors for different types of numbers.

Your Turn

Practice: Complete the solution to the example problem for 1000 envelopes.

Reflect: How does this solution relate to the idea that square numbers have an odd number of factors? ■



The NCTM *Principles and Standards for School Mathematics* affirm the classroom role of the number theory ideas in Sections 4.1 and 4.2 with statements like these.

- Students (in grades 3–5) should recognize that different types of numbers have particular characteristics: for example, square numbers have an odd number of factors and prime numbers have only two factors (p. 151).
- For students (in grades 6–8) p Tasks p involving factors, multiples, prime numbers, and divisibility, can afford opportunities for problem solving and reasoning (p. 217).

Problems and Exercises for Section 4.1

A. Reinforcing Concepts and Practicing Skills

- How can you decide whether 7 is a factor of 87?
- List the factors of 24.
- To find the factors of 1225, what is the largest number you would have to test? What theorem assures you of that?
- Use a calculator to find the factors of each number.
 - 78
 - 156
 - 252
- List a factor and a multiple of 253 (other than 1 and 0).
- Explain why 0 is a multiple of every whole number.
- Show in two different ways that 13 is a factor of 299.

For Exercises 8–15, fill in the blank with “factor” or “multiple.”

- 8 is a ___ of 64.
- 9 is a ___ of 3.
- 0 is a ___ of every natural number.
- 1 is a ___ of every natural number.
- A ___ of a non-zero number is always less than or equal to the number.
- A non-zero ___ of a number is always greater than or equal to the number.
- A number $6n^2$ is a ___ of the number $2n$.
- A number is a ___ of its cube.
- List some multiples of the number $3n$.
- How can you quickly decide whether a number is divisible by
 - 2?
 - 3?
 - 4?
 - 5?
 - 6?
- Which of the following are divisible by 2? By 3? By 4? By 5? By 6?
 - 699
 - 1320
 - 2645
 - 5736
- If a number n is divisible by 24, what are some other numbers n is divisible by?

In Exercises 20–22, fill in the missing digits so the number will be divisible by 3.


- 4 _ 3 _
- _ 7 _ _
- 72 _ 94

In Exercises 23–25, fill in the missing digits so the number will be divisible by 6.

- 5 _ 4 _
- 24 _ 6 _
- 82 _ 94

In Exercises 26–28, fill in the missing digits so the number will be divisible by 9.

- 8 _ 3 _
- 73 _ 5 _
- 62 _ 95
- Decide if the statement is true or false. Tell why.
 - $4 \mid 20$
 - 12 is a factor of 6
 - $24 \mid 24$
 - $0 \mid 8$
 - $4 \mid 0$
 - 60 is a multiple of 15
 - 24 is a divisor of 8
 - Every number is divisible by itself.

- Show that 6 is a perfect number.
- Show that 15 is a deficient number.
- Show that 18 is an abundant number.
- What is the smallest even, abundant number? Explain.
- Give examples to show the following.
 - A square number has an odd number of factors.
 - There are both even and odd square numbers.
 - A number can be even, square, and abundant.
- A chewing gum factory packages five sticks of gum into a small pack and three small packs into a large pack. A total of 43,860 sticks of gum are produced in a unit of time. Which divisibility rules can help you decide whether a whole number of full large packs of gum, with none left over, will be produced? Will a whole number of packages be produced?
- How could you quickly decide whether 1516 people could be seated at a banquet 4 at a table with all full tables and no extra people? Can they?
- Can 114 days be divided into a whole number of 7-day weeks?
- Describe the different ways a designer could lay out parking lot spaces for 84 cars in rows, with an equal number of spaces in each row and no spaces left over.
- Suppose that you were asked to introduce the number 28 to an audience. Write a paragraph telling as many things as possible about 28. Use as many of the terms defined in this chapter as you can.
-  A 73-year-old numerologist claimed that when she formed an 8-digit number by writing the year she was born twice in succession, it was divisible by her age. She was born in 1927. Use your calculator to verify that she was correct. Was this unusual, or could any other 73-year-old person say the same? Give examples to illustrate your answer.
- To be a leap year, a year must be divisible by 4. However, some years that are divisible by 4, such as any year that is a multiple of 100, need to be checked further to see if they are leap years. Such a year will be

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a leap year only if it is also divisible by 400. For example, 1300 (a multiple of 100) is divisible by 4, but is not a leap year because it is not divisible by 400. Which of the following years are leap years? Why or why not?

- a. 700 b. 1992 c. 1776
d. 2000 e. 2010

B. Deepening Understanding

42. a. If a number is divisible by 10, is it necessarily divisible by 5? b. If a number is divisible by 5, is it necessarily divisible by 10? Explain.
43. The following conjectures were made by students. Which do you think are false? Give a counterexample for each conjecture if possible.
- Only half of the non-zero even numbers up to 100 are divisible by 4.
 - A number is divisible by 8 if it is divisible by 4 and divisible by 2.
 - If 12 divides a number, 6 also divides the number.
 - If a number is divisible by 4 and 6, it must be divisible by 24.
44. Describe the numbers that have exactly three factors by looking for a pattern.
45. Decide whether each statement is *sometimes*, *always*, or *never* true.
- A natural number has an unlimited number of multiples.
 - A natural number has an even number of factors.
 - With natural numbers other than 1, if a is a multiple of b , then a is a factor of b .
 - An odd number is a factor of an even number.
 - When n is a factor of a number, all factors of n are also factors of the number.
46. Try to find a counterexample for each, where n , a , and b are natural numbers. Give a numerical example for those that you think are true.
- If $n \mid ab$, then $n \mid a$ or $n \mid b$.
 - If $n \mid (a + b)$, then $n \mid a$ and $n \mid b$.
 - If $n \mid a$ and $n \mid b$, then $n \mid (a + b)$.
 - If $n \mid a$ and $a \mid b$, then $n \mid b$.
 - If $n \mid (a + b)$, then $n \mid a$ or $n \mid b$.
 - If $n \mid a$ and $n \mid b$, and $a > b$, then $n \mid (a - b)$.
47. Show that 220 and 284 are amicable numbers.
48. The next perfect number greater than 28 is 496. Prove that it is a perfect number.
49. Which numbers between 30 and 40 are deficient? Which are abundant?
50. Make a table showing what happens when you add two even numbers, two odd numbers, and an odd and an even number. Repeat for multiplication.
51. A number that is the sum of some of its factors, such as $12 = 6 + 4 + 2$, has sometimes been called a *semiperfect* number. Find two more semiperfect numbers under 30 and prove that they are semiperfect.
52. A shopper had quarters and dimes and wanted to have the exact amount for a \$4.56 purchase. Is that possible? Use divisibility to explain your answer.
53. A certain number has a remainder of 1 when divided by 2, 3, 4, and 5, but when divided by 7 the remainder is 0. What is the number? Is there more than one correct answer? Explain.
54. A sports shop sold tennis balls only in full boxes of 3 or in full boxes of 6. The shop's records showed that 5717 tennis balls had been sold during the past month. How did the owner know that the records were incorrect?

**55. The House Numbers.**

- Bill was generally superstitious and wondered whether his house number, 156, was deficient or abundant! The proper factors of 156 are 2, 3, 4, 6, 12, 13, 26, 39, 52, and 78. State how Bill can estimate the answer, without adding up all the factors.
- Janie's house number is 273, with proper factors of 3, 7, 13, 21, 39, and 91. How can Janie estimate, without adding all the numbers, whether her number is deficient or abundant?

C. Reasoning and Problem Solving

56. The following are some interesting theorems about number relationships. Select specific numbers and give an example for each theorem.
- Every 6-digit number with identical ones and thousands periods, such as 325,325, is divisible by 13.
 - Every 4-digit palindrome is divisible by 11. (A palindrome number reads the same backward or forward, such as 2332.)
 - Every even number greater than 46 is the sum of two abundant numbers.
 - If the digits of any 2-digit number are reversed and the numbers subtracted, the difference is a multiple of 9.
57. Use numerical examples to illustrate that, if $n \mid (a + b)$ and $n \mid b$, then $n \mid a$.
58. A student generalized that "If a number is divisible by 2 and by 5, then it is divisible by 10." Choose the appropriate divisibility theorem, and use it to verify this generalization.
59. To verify the theorem about a divisibility test for 4, a teacher began her reasoning as follows:
When you divide a number by 100, the remainder is the last two digits of the number. For example, $724 \div 100 = 7$,

Remainder 24. We can write this as $724 = 100(7) + 24$. So if n is any number with three digits or more, we can write $n = 100q + r$, where q is the quotient when n is divided by 100 and r is the number formed by the last two digits of n . Since $4 \mid 100q$, the number n will be divisible by 4 if it is also true that _____.

Complete this last statement, and give a divisibility theorem that supports your reasoning.

60. Use reasoning similar to that in Exercise 59 to verify the theorem on divisibility by 8.
61. Give a numerical example to illustrate the following theorem and then use the definition of divisibility to verify it.
For natural numbers a , b , and c , if $a \mid b$ and c is any natural number, then $a \mid bc$.
62. The fraction $\frac{16}{64}$ is interesting because you can use the *incorrect* procedure of crossing out a digit in the numerator and a digit in the denominator to get an equivalent fraction:

$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}.$$

Find three other fractions that have this characteristic. (*Hint:* The fractions have 2-digit numerators and denominators, with common factors of 13, 19, and 7, respectively.)

63. **The Age-Guessing Contest Problem.** A mother and daughter entered an age-guessing contest. The mother announced that she was about three times as old as her daughter and that their ages were alike in that each age left a remainder of 3 when divided by 4, a remainder of 2 when divided by 3, and a remainder of 1 when divided by 2. What were the ages of the daughter and the mother?
64. **The License Plate Number Problem.** An eccentric mathematics teacher ordered a 6-digit license plate with a number on it that contained all the digits 1 through 6 and met the following conditions. When read from left to right, its first two digits formed a

number divisible by 2, its first three digits formed a number divisible by 3, and so on, ending with a complete 6-digit number divisible by 6. What was the teacher's license plate number?

65. **The Football Score Problem.** Bob's mother called her brother and reported that Bob's football team held the other team scoreless and made all its points on the touchdowns (6 points) and field goals (3 points), missing every point after the touchdown kick. She told him that the final score was 43–0. Her brother told her that this couldn't be the correct score. How did he know?

66. **The House Number Problem.** Cindy posed the following problem to her friend Roger.

Cindy: Can you figure out my house number? It's an even 3-digit number. Each digit represents the age of one of my three children. The product of the digits is 72 and the sum of the digits is 14.

Roger: Hmm p I'm not sure.

Cindy: Oh, yes, my oldest child likes chocolate ice cream.


Roger: I have it!

What is Cindy's house number?

67. **The Lost Receipt Problem.** A secretary lost a receipt for 24 stenographer's notebooks that he had purchased for the office. He remembered that the middle two digits of the total 4-digit dollars and cents cost were 7 and 3. That made the total \$7.3?, with the question marks representing the missing digits. He also remembered that the cost of each notebook was between \$2 and \$3. How much did each notebook cost?

D. Communicating and Connecting Ideas

68. Choose some numbers and work in a small group to figure out how to apply the tests for divisibility by 4, 6, 7, 8, 9, and 11 given on pp. 197, 198, and 199.

 a. Use a calculator to select some numbers that are and some that aren't divisible by these numbers. Give an example to show that each test works.

b. Write an answer to the question, "Which tests, if any, are interesting but too complicated to be of much practical value in checking divisibility?" Present your conclusions and reasons for your choices.

69. **Historical Pathways.** Marin Mersenne (1588–1648) formulated the problem of finding *multiply perfect* numbers, in which the sum of the proper factors of a number is a multiple of the number itself. Show that 120 and 672 are multiply perfect numbers.

70. **Making Connections.** How do the ideas of divisibility and multiples relate to the study of fractions?



Section 4.2 | PRIME AND COMPOSITE NUMBERS

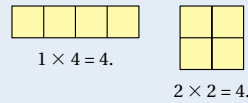
- Defining Prime and Composite Numbers
- Techniques for Finding Prime Numbers
- The Role of Prime Numbers in Mathematics
- Greatest Common Factor and Least Common Multiple
- Prime and Composite Numbers and Relationships

In this section, we use the concept of factors of a number to develop the ideas of prime and composite numbers. We illustrate why prime numbers are considered *building blocks* for natural numbers and develop ways to find the prime factorization of any natural number. Prime factorization is then used to find the greatest common factor and least common multiple of two numbers. Finally, we explore some interesting patterns and relationships involving prime numbers. Mini-Investigation 4.2 asks you to consider members of a special set of numbers that will play a major role in this section.

Draw pictures on graph paper to illustrate that the numbers 8, 9, 10, and 12 can be shown by more than just a single-row rectangle.

Mini-Investigation 4.2 Making a Connection

Which numbers from 2 to 30 can be shown only by a single-row rectangle? Note, for example, that the number 4 can be shown by both a single-row rectangle and a double-row rectangle.



Defining Prime and Composite Numbers

In Mini-Investigation 4.2, you classified the numbers 2 through 30 into two sets, those that must be shown by a single-row or column rectangle (2, 3, 5, 7, 11, ...) and those that can be shown by a rectangle with more than one row of squares (4, 6, 8, 9, 10, 12, ...). The first five models for these numbers are shown in Figure 4.3.

Factors of the modeled numbers may be determined by looking at the length and the width of the rectangles. Note that the single-row numbers have exactly two factors, the number itself and 1. The multiple-row numbers have more than two factors. This classification suggests the following definition.

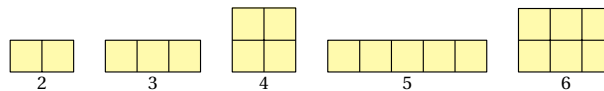


FIGURE 4.3 | Geometric models for five of the numbers 2–30.

Definition of Prime and Composite Numbers

A natural number that has exactly two distinct factors is called a **prime number**.
A natural number that has more than two distinct factors is called a **composite number**.

Therefore the prime numbers are the numbers that may be shown only as a single-row rectangle and that have exactly two factors. The composite numbers may be shown by a rectangle that has more than one row of squares and thus have more than two factors. For example, the number 5 is a prime number because it has exactly two factors, 1 and 5. The number 6 is a composite number because it has more than two factors, 1, 2, 3, and 6.

©Tony Stone Images

Radio telescopes might send prime number sequences into space to communicate with intelligent beings in other galaxies.

The number 1 has only *one* distinct factor, so it is neither prime nor composite. This classification of the natural numbers greater than 1 into the two sets, prime and composite, gives us a useful language for discussing number properties and relationships. Exercise 45 at the end of this section will help you consider some common misconceptions about prime numbers.

Techniques for Finding Prime Numbers

Mini-Investigation 4.3 gives you an opportunity to use and analyze a method for identifying prime numbers that the Greek mathematician, Eratosthenes (200 B.C.), developed. It is called the **Sieve of Eratosthenes**.

Connection to the PreK–8 Classroom

A Sample Student Page: Representing Prime and Composite Numbers

The NCTM *Principles and Standards for School Mathematics* focus on the important role of representation in school mathematics learning. Some excerpts from the standards for grades 3–5 illustrate this focus. “Teachers can and should emphasize the importance of representing mathematical ideas in a variety of ways Many students need support in constructing pictures, graphs, tables, and other representations Organizing work in this way highlights patterns...” Study the page from *Scott-Foresman–Addison Wesley Math*, Book 5 (1998) shown in Figure 4.4 and answer the questions to see the importance of representations in developing number theory ideas.

- **What** number theory ideas are developed here?
- **What** different types of representations are used?
- **Comment** on the role of models and looking for patterns.

Chapter 4
Lesson 14

Exploring Prime and Composite Numbers


Problem Solving Connection

- Look for a Pattern
- Use Logical Reasoning

Materials
hundred chart

Vocabulary
prime number
whole number with exactly two different factors
composite number
whole number greater than 1 that is not prime

Did You Know?
A computer has been used to find a prime number that, if printed in a newspaper, would fill 12 pages.



Explore

Eratosthenes (ehr uh TAFS ruh seez) was a mathematician from Cyrene (now known as Libya) who lived about 2,200 years ago. He studied **prime numbers**. The number 3 is an example of a prime number. It has only two factors, 1 and 3.

Composite numbers have more than two factors. An example of a composite number is 8. Its factors are 1, 2, 4, and 8.

1×8 ●●●●●●●● 2×4 ●●●●

You tell whether a number is prime by testing possible factors. There are no other factors for 7 besides 7 and 1. So, 7 is a prime number.

1×7 ●●●●●●●●

Eratosthenes' Sieve is a process that "strains" out composite numbers and leaves prime numbers behind. Use it to find the prime numbers between 1 and 100.


Work Together

1. Use a hundred chart. Follow the directions to cross out composite numbers and circle prime numbers.
 - a. Cross out 1. It has only 1 factor.
 - b. Circle 2, the least prime number. Cross out all numbers divisible by 2.
 - c. Repeat this step with 3, the next prime number.
2. Continue this process until you reach 100.
3. List all of the circled numbers. There should be 25.

X	2	3	X	5	X	7	X	9	10
11	12	13	14	15	16	17	18	19	20
21	22								

Talk About It

What pattern can you see in the list of prime numbers?



Cyrene was the capital of ancient Cyrenais.

FIGURE 4.4 | Excerpt from *Scott Foresman–Addison Wesley Math*.
 (Source: Charles, Randy, et al. *Scott Foresman–Addison Wesley Math*, Book 5, Addison Wesley Longman, Inc., 1998, p. 204, used with permission.)

Write a description of what you know about the factors of the numbers crossed out and those not crossed out.

Mini-Investigation 4.3 Finding a Pattern

What do you discover about the numbers that remain in the following list after you cross out all the multiples of 2 other than 2 and do the same for 3, 5, and 7?

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

The method presented in Mini-Investigation 4.3 identifies prime numbers in a list of consecutive natural numbers. Each time multiples of a number are crossed out, numbers that have that number as a factor are eliminated. Continuing this process eliminates all numbers that have more than two factors, leaving the prime numbers. This method identifies prime numbers from a consecutive list, but it isn't particularly useful in answering a question such as, "Is the number 323 prime?" To answer such a question, we would probably use a calculator to identify the factors of the number, as in Example 4.6.

Example 4.6

Using a Calculator to Identify Prime Numbers

Use a calculator to decide whether 323 is a prime number.

Solution

Janell's thinking: Here's how I used a scientific calculator. I entered 323 into memory.

Enter	Push
323	

I got the square root of 323. Because of the factor test theorem, I need test only factors through 17.

Push	Display	Push	Display
	323		17.972201

I tested all necessary factors and noted that 17 is a factor of 323.

Push	Display	Push	Enter	Push	Display
	323		17		19

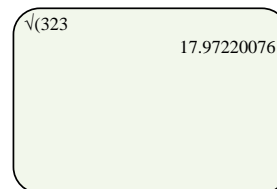
The number 323 isn't prime.

Tom's thinking: I used the table feature of a graphing calculator. First, I calculated the square root of 323 in order to determine the divisors to be checked, by entering the following.

Key Sequence

3 3

Display



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Next, I entered the function $Y1 = 323/x$ under the $Y=$ menu so that the calculator would divide 323 by successive values of x .

Key Sequence

$Y=$ 3 2 3 \div X ENTER

Display

Plot1 Plot2 Plot3
 $Y1=323/X$ ■
 Y2=
 Y3=
 Y4=

Then I used the table setup menu to set the calculator up to start with an initial value of 1 and to increase the size of x by 1 each time.

Key Sequence

2nd [TBLSET]
 1 ▾
 1

Display

TABLE SETUP
 TblStart=0
 Δ TBL=1 ■
 Indpnt: Auto Ask
 Depend: Auto Ask

Finally, I displayed the table and used the arrow keys to scroll the Y1 column to search for whole-number results.

Key Sequence

2nd [TABLE]

Display

X	Y1
1	323
2	161.5
3	107.67
4	80.75
5	64.6
6	53.833
$\bar{X}=1$	

X	Y1
12	26.917
13	24.846
14	23.071
15	21.533
16	20.188
17	19
$\bar{Y1}=19$	

When the divisor is 17, the quotient is exactly 19. Therefore 323 is not a prime number; it has factors other than 1 and itself.

Your Turn

Practice: Use a calculator to decide whether 431 is a prime number.

Reflect: After you tested the numbers 2, 3, and 5, did you need to test any multiples of these numbers? Explain. ■

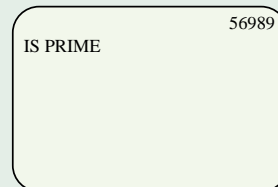
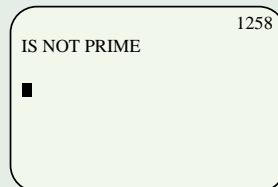


Spotlight on Technology Graphing Calculator Programs

You have used a graphing calculator to do several different types of calculations. Now you will learn how to program graphing calculators to carry out repetitive calculations to complete many other tasks. In this feature, we focus on *programming a graphing calculator to determine whether a given number is prime*.

Follow the instructions provided to enter the program into one type of graphing calculator, a TI-73. With modest adaptation, the program can be used in many other types of graphing calculators.

- Press the keys `PRGM` `▶` `▶` `ENTER` to name your new program.
- Press `2nd` `[TEXT]` to display the keyboard for typing. Use the arrow keys and `ENTER` to select the letters from the alphabet table to type in your program name, for example, PRIMES.
- When you finish typing your program name, arrow to highlight `DONE` and press `ENTER` to finish typing. Press `ENTER` a second time to get into the program editor. Note that the screen shows your program name and a flashing cursor to the right of a colon (:). This display indicates you now can type in your program.
- Following these instructions, each line of the program PRIMES is shown, along with information on how to access the commands. *After you complete a line, always press `ENTER` to move to the next line. You do not need to type in a colon; each one is automatically placed after you hit `ENTER`.*
- After entering the program PRIMES, run it. Press `PRGM` and then use the down-arrow key to highlight PRIMES. Press `ENTER` `ENTER` to start the program. The screen to the left shows that 1258 is not prime and the screen to the right shows that 56,989 is prime. Can you find a prime number larger than 56,989?



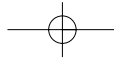
Program for Deciding Whether a Number Is Prime

```

: INPUT "NUMBER=" ,N
: FOR (K, 2, √(N), 1)
: IF N/K = ROUND(N/K, 0)
: GOTO 1

```

`PRGM` `▶` `1` `2nd` `[TEXT]` and type "NUMBER=",N
 Finish typing by highlighting `DONE` and pressing `ENTER`
 Press `PRGM` `4` to enter the **for**(command.
 Press `2nd` `[TEXT]` to enter `K,2,√(N),1`
 Press `PRGM` `1` to enter **if**, `2nd` `[TEXT]` to enter `N/K`, and exit with `DONE` `ENTER`. Press `MATH` `▶` `2` to enter **round**(. Press `2nd` `[TEXT]` to enter `N/K,0` and exit the keyboard with `DONE` and `ENTER`.
`PRGM` `0` `1`



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:END	PRGM	7	
:CLRscreen	PRGM	▶	8
:DISP N,"IS PRIME"	PRGM	▶	3
		2nd	[TEXT] N,"IS PRIME" DONE
	ENTER		
:GOTO 2	PRGM	0	2
:LBL 1	PRGM	9	1
:CLRscreen	PRGM	▶	8
:DISP N,"IS NOT PRIME"	PRGM	▶	3
		2nd	[TEXT] N,"IS NOT PRIME"
	DONE	ENTER	
:LBL 2	PRGM	9	2
:PAUSE	PRGM	8	
:STOP	PRGM	▲	▲
		▲	ENTER

The Role of Prime Numbers in Mathematics

Prime Numbers as Building Blocks. The prime numbers are building blocks for composite numbers, as you can see by breaking a composite number such as 30 into a product of prime factors. Factor trees, three of which are shown in Figure 4.5, graphically show this process. Even though three different factor trees are shown, the factors in the bottom row are the same, regardless of the order in which they appear. In each factor tree, the product of factors for 30 consists of 2, 3, and 5 multiplied in some order.

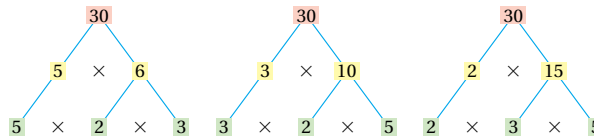


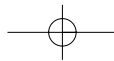
FIGURE 4.5 | Factor trees for the number 30.

The three factor trees shown suggest the following important theorem, which Euclid included in his book *Elements* in 320 B.C.

**Fundamental Theorem of Arithmetic
(Unique Factorization Theorem)**

Each composite number can be expressed as the product of prime numbers in exactly one way, disregarding the order of the factors.

This theorem guarantees that each composite number is completely identified by a unique product of prime numbers, which might be considered its “fingerprint.” Two different composite numbers cannot equal the same product of primes, and two different products of primes cannot equal the same composite number. The number 1 is defined as neither a prime number nor a composite



number, because if 1 were one of these types of numbers, we could write $6 = 1 \times 2 \times 3$ and $6 = 1 \times 1 \times 2 \times 3$, and the prime factorization of 6 would not be unique.

Finding the Prime Factorization of a Number. When a number is expressed as a product of primes, the expression is called the **prime factorization** of the number. For example, $42 = 2 \times 3 \times 7$, so $2 \times 3 \times 7$ is the prime factorization of 42. When prime factors repeat in a prime factorization, we represent them with *exponents*. For $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$, we write $360 = 2^3 \times 3^2 \times 5$.

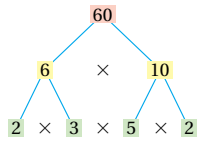


FIGURE 4.6 | The prime factorization of $60 = 2^2 \times 3 \times 5$.

Factor trees, such as those in Figure 4.5, can be used to record the process for finding the prime factorization of a number. For example, to find the prime factorization of 60, begin by breaking 60 into the product of two factors. Then break each of those factors into the product of two factors, and so on, until the numbers in the bottom row of the factor tree are prime and the prime factorization of 60 appears. This process for finding the prime factorization of 60 is depicted in Figure 4.6.

A *stacked division* procedure, based on the fundamental theorem of arithmetic, can also be used to find the prime factorization of 60. These two procedures show that $60 = 2^2 \times 3 \times 5$. It may also be used to find the prime factorization of a number.

$$\begin{array}{r} 5 \\ 3 \overline{)15} \\ 2 \overline{)30} \\ 2 \overline{)60} \end{array}$$

To use the division method for finding prime factorizations, choose the smallest prime that will divide the number. Divide by the prime and determine whether this prime will also divide the quotient. Repeat with successive quotients as long as possible. When this prime will not divide a quotient, use the next largest prime and continue the process. When you reach a quotient that is prime, you can see—in the spirit of the Frank and Ernest cartoon in Figure 4.7—a string of prime numbers, the quotient and divisors, that can be multiplied to produce the original number. The process essentially divides prime factors out of the number and uses these factors as the prime factorization. Example 4.7 illustrates how and then asks you to find the prime factorization of a number.

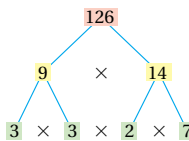
Example 4.7

Finding the Prime Factorization

Find the prime factorization of 126.

Solution

Dana's thinking: I just made a factor tree for 126. The bottom row of the factor tree is the prime factorization.



Jordan's thinking: I thought, 126 is 2 times 63, but 63 is 9 times 7 or 3 times 3 times 7. Then I wrote the equation $126 = 2 \times 3^2 \times 7$.

Wesley's thinking: I used the following division method.

$$\begin{array}{r}
 7 \\
 3 \overline{)21} \\
 3 \overline{)63} \\
 2 \overline{)126}
 \end{array}$$

The prime factorization of 126 is $2 \times 3^2 \times 7$.

Your Turn

Practice: Write the prime factorizations of 56, 150, and 252. Use exponents.

Reflect: How do you know that the answer to the example problem is the only prime factorization of 126? ■

Greatest Common Factor and Least Common Multiple

The ideas about sets you learned in Chapter 2 and the idea of the prime factorization of a number can be used to help find two important numbers—the greatest number that is a factor of two given numbers and the smallest number that is a multiple of two given numbers. These procedures will be developed in following subsections.

Greatest Common Factor. By listing all factors you could conclude that 16, for example, is the greatest number that is a factor of both 64 and 48. This conclusion suggests the following definition of a useful idea in elementary number theory.

Definition of Greatest Common Factor

The **greatest common factor (GCF)** of two natural numbers is the greatest natural number that is a factor of both numbers.

Note that the greatest common factor is sometimes called the greatest common divisor, and in that case is represented by GCD.

Using the Intersection of Sets to Find the Greatest Common Factor. To find the GCF when dealing with a pair of numbers that have a reasonably small number of factors, you can just list the factors of each number, identify the common factors, and select the greatest of these common factors. The Venn diagram in Figure 4.8 shows how this could be done.

This can be expressed using set notation as follows.

A = the set of factors of 24 = {1, 2, 3, 4, 6, 8, 12, 24}.

B = the set of factors of 30 = {1, 2, 3, 5, 6, 10, 15, 30}.

$A \cap B$ = the set of common factors of 24 and 30 = {1, 2, 3, 6}.

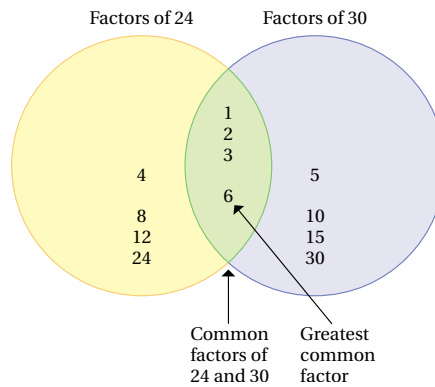


FIGURE 4.8 | Intersection of factor sets method of finding the GCF.

The greatest number in the set of common factors, 6, is the GCF of 24 and 30.

Using Prime Factorization to Find the GCF. A method for finding the GCF that is more efficient than listing factors, especially when dealing with larger numbers, involves writing the prime factorization of each number. First, write the prime factorization of each number.

$$24 = 2 \times 2 \times 2 \times 3$$

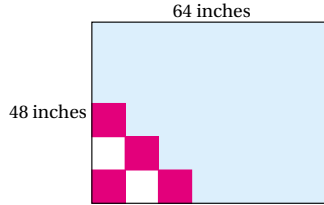
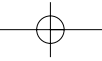
$$30 = 2 \times 3 \times 5$$

Since 2×3 is the largest factor contained in both of the prime factorizations, 2×3 , or 6, is the GCF of 24 and 30. In Example 4.8 you can use the GCF to solve a practical problem.

Example 4.8

Problem Solving: The Largest Square Tile

What is the largest size of square tile that could be used in making a mosaic that is 64 inches long and 48 inches wide?



Solution

Kiree's thinking: To solve the problem, I need to find the GCF of 48 and 64. The factors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48. The factors of 64 are 1, 2, 4, 8, 16, 32, and 64. Because 16 is the largest number in both lists, it's the GCF. The largest possible tile is 16 inches square.

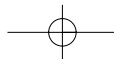
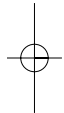
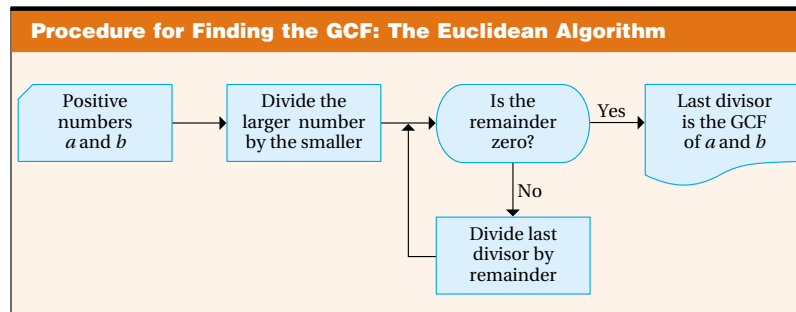
Tavin's thinking: I used prime factorization and started with $48 = 2 \times 2 \times 2 \times 2 \times 3$, and $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$. I discovered that $2 \times 2 \times 2 \times 2$ is contained in both prime factorizations, so the GCF of 48 and 64 is $2 \times 2 \times 2 \times 2$, or 16. The largest possible tile is 16 by 16.

Your Turn

Practice: Suppose the mosaic in the example is 60 inches long and 45 inches wide. What is the largest square tile that could be used?

Reflect: Which of the two methods in the example problem seems easier? Explain. ■

The **Euclidean algorithm** is the name given to a method that can be used with a calculator to find the GCF of a pair of larger numbers. The GCF of a pair of numbers is the same as the GCF of the smaller of the two numbers and the remainder when the larger is divided by the smaller. For example, $\text{GCF}(30, 24) = 6$, and $\text{GCF}(24, 6) = 6$. This idea is the basis for the Euclidean algorithm, as described by the following procedure.



The Euclidean algorithm allows you to reduce the size of the numbers continually until you can easily find the GCF. Let's use the flow chart and the following divisions to help explain how to find the GCF of 84 and 308.

$$\begin{array}{l} \text{a) } \begin{array}{r} 3 \\ 84 \overline{)308} \\ \underline{252} \\ 56 \end{array} \qquad \text{b) } \begin{array}{r} 1 \\ 56 \overline{)84} \\ \underline{56} \\ 28 \end{array} \qquad \text{c) } \begin{array}{r} 2 \\ \textcircled{28} \overline{)56} \\ \underline{56} \\ 0 \end{array} \end{array}$$

In Equation a, we divide the larger of the two numbers, 308, by the smaller, 84. The remainder, 56, isn't zero, so use it as the new divisor. In Equation b, we divide the last divisor, 84, by the remainder, 56. The remainder, 28, isn't zero, so use it as the new divisor. Finally, in Equation c, we divide the last divisor, 56, by the remainder, 28. The remainder is zero, so the last divisor, 28, is the GCF of 84 and 308.

Example 4.9 illustrates how the integer division feature on a calculator can be used along with the Euclidean algorithm to find the GCF of a pair of larger numbers.

Example 4.9

Using a Calculator to find GCF

Use a calculator that divides integers and shows remainders and the Euclidean algorithm method to find the GCF of 475 and 1501.

Solution

Key Sequence	Display
1501 <input type="button" value="INT ÷"/> 475 <input type="button" value="="/>	3 76
475 <input type="button" value="INT ÷"/> 76 <input type="button" value="="/>	6 19
76 <input type="button" value="INT ÷"/> 19 <input type="button" value="="/>	4 0

The GCF of 1501 and 475 is 19.

Your Turn

Practice: Use a calculator with an key to find the GCF of 2940 and 1260.

Reflect: How could you use a calculator *without* an key to find the GCF of 2599 and 2825? ■

Least Common Multiple. It is useful to be able to find the smallest number that is a multiple of each of two given numbers. Suppose the two numbers are 9 and 12. By listing multiples of each, you could conclude that 36 is the smallest number that is a multiple of both 9 and 12. This conclusion suggests the following definition of a useful idea in elementary number theory.

Definition of Least Common Multiple

The **least common multiple (LCM)** of two natural numbers is the smallest natural number that is a multiple of both the natural numbers.

Two useful methods for finding the least common multiple of two numbers will be described in the following subsections.

Using Intersection of Sets to Find the Least Common Multiple. When dealing with a pair of small numbers, you can just list some of the multiples of each number, identify the common multiples, and select the smallest non-zero common multiple. The Venn diagram in Figure 4.9 shows this procedure.

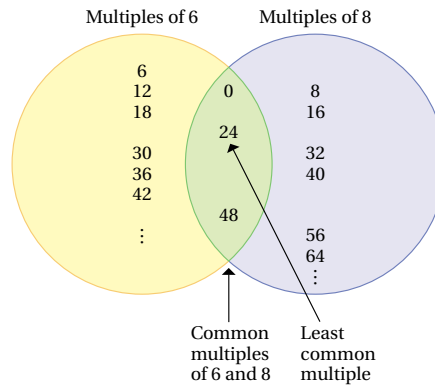


FIGURE 4.9 | Intersection of sets of multiples method for finding the LCM.

This can be expressed using set notation as follows.

$$C = \text{the set of multiples of } 6 = \{0, 6, 12, 18, 24, 30, 36, 42, 48, \dots\}$$

$$D = \text{the set of multiples of } 8 = \{0, 8, 16, 24, 32, 40, 48, 56, 64, \dots\}$$

$$C \cap D = \text{the set of multiples of } 6 \text{ and } 8 = \{0, 24, 48, \dots\}$$

The smallest common multiple, 24, is the LCM of 6 and 8.

Using Prime Factorization to Find the Least Common Multiple. A method for finding the LCM that is more efficient than listing multiples, especially when dealing with larger numbers such as 28 and 30, involves writing the prime factorization of each number. First, write the prime factorization of each number.

$$28 = 2 \times 2 \times 7$$

$$30 = 2 \times 3 \times 5$$

Any non-zero multiple of 28 must at least have the factors 2, 2, and 7. Any non-zero multiple of 30 must at least have the factors 2, 3, and 5. Using these prime factors the largest number of times it appears in either of the prime factorizations, we find that the smallest multiple that has the necessary factors to be a multiple of both 28 and 30 is $2 \times 2 \times 3 \times 5 \times 7$, or 420. The LCM of 28 and 30 is 420.

Example 4.10

Problem Solving: Orbiting Spacecrafts

Two spacecrafts have elliptical orbits around the earth. Spacecraft Alpha makes one complete orbit in 90 minutes. Spacecraft Beta makes one complete orbit in 120 minutes. At this moment, the spacecrafts are beside each other in their orbits. In how many minutes will they be beside each other again?

Solution

Regina's thinking: The multiples of 120 are 120, 240, 360, etc. Multiples of 90 are 90, 180, 270, 360, etc., so the LCM of 120 and 90 is 360. The satellites will be in the same location after 360 minutes.

Vito's thinking: I used prime factorization and started with $120 = 2 \times 2 \times 2 \times 3 \times 5$, and $90 = 2 \times 3 \times 3 \times 5$. The LCM has to have three 2's, two 3's, and a 5 in it, so $2 \times 2 \times 2 \times 3 \times 3 \times 5$ works. The LCM of 120 and 90 is 360. It will take 360 minutes for the satellites to be in the same location.

Your Turn

Practice: Find the LCM of 9 and 12; 12 and 18; 28 and 42.

Reflect: Which of the students' thinking in the example solution makes the most sense to you? Explain why. ■

The GCF and LCM of two numbers can easily be found when using exponent notation in the prime factorization of the two numbers. For example $12 = 2^2 \times 3^1$, and $18 = 2^1 \times 3^2$. The GCF is found by using the *minimum* exponent for each prime power in the prime factorization of the numbers, so $\text{GCF}(12, 18) = 2^1 \times 3^1$, or 6. The LCM, on the other hand, is found by using the *maximum* exponent for each prime power in the prime factorizations, so $\text{LCM}(12, 18) = 2^2 \times 3^2$, or 36.

Exercise 50 at the end of this section gives another interesting procedure for finding the LCM of two numbers.

Connection to
the PreK–8
Classroom

The NCTM *Principles and Standards for School Mathematics* state that “In grades 3–5 students should have learned to generate equivalent forms of fractions . . .” (p. 215). “Middle grades students should continue to refine their understanding of addition, subtraction, multiplication, and division as they use these operations with fractions . . .” (p. 218). The ability to find the GCF of two numbers is used to find an equivalent, lowest-term representation for a fraction. Also, the LCM is used to find the least common denominator when adding fractions. Both ideas are used in algebra to simplify rational expressions.

Prime and Composite Numbers and Relationships

Patterns and relationships involving prime and composite numbers have always fascinated mathematicians and students alike. In this subsection, we present some of the interesting relationships involving LCM and GCF, as well as relationships involving prime numbers.

Relationships Involving GCF and LCM. As you complete Table 4.4, you might discover an answer to the simple question “How is the GCF related to the LCM?”

TABLE 4.4 | Discovering a Relationship between the GCF and the LCM

Pair of Numbers a, b	$a \times b$	GCF (a, b)	LCM (a, b)	GCF (a, b) \times LCM (a, b)
6, 8	48	2	24	48
6, 10	60	2	30	60
8, 9	72	1	72	72
9, 12	?	?	?	?
9, 15	?	?	?	?
12, 20	?	?	?	?

Using inductive reasoning and the table, you might generalize the following theorem.

Theorem: The GCF–LCM Product

The product of the GCF and the LCM of two numbers is the product of the two numbers.

The above theorem and the idea that numbers a and b are **relatively prime** if and only if $\text{GCF}(a, b) = 1$ can be used to answer other interesting relationship questions, such as the one in Exercise 42 at the end of this section. Example 4.11 applies the GCF–LCM theorem.

Example 4.11

Using the GCF–LCM Product Theorem

Find the GCF and LCM of 888 and 259.

Solution

Ryan's thinking: I first used the Euclidean algorithm and a calculator with the integer divide feature to find the GCF.

Key Sequence Display

888 259
 259 111
 111 37

The GCF of 888 and 259 is 37. Using the GCF–LCM theorem, I then found the LCM.

$$\begin{aligned} \text{GCF}(888, 259) \times \text{LCM}(888, 259) &= 888 \times 259 = 229,992 \\ 37 \times \text{LCM}(888, 259) &= 229,992 \\ \text{LCM}(888, 259) &= 229,992 \div 37 = 6216 \end{aligned}$$

Lauren's thinking: I used a fraction calculator to find the GCF. Then I used the GCF–LCM theorem to find the LCM. I first entered the two numbers as a fraction into my calculator.

Key Sequence Display

259 888 259/888

Now I use the function to reduce the fraction.

Key Sequence Display

 7/24

When I press , the common factor appears that was used to reduce the fraction.

Key Sequence Display

 37

(If the lower left corner of my calculator screen shows **N/D** → **n/d**, indicating that 37 isn't the GCF, I would need to use the again and multiply common factors used for reducing to find the GCF.)

In this case, 37 is the GCF of 888 and 259. The LCM is $(888 \times 259) \div 37$, or 6216.

Your Turn

Practice: Find the GCF and LCM of

- a. 1501 and 475.
- b. 2599 and 2825.

Reflect: Explain in your own words how to find LCM (a, b) easily when you know GCF (a, b). ■

Relationships and Patterns Involving Prime Numbers. One interesting prime number topic is the frequency of occurrence of consecutive primes with a difference of 2, such as 3 and 5, which are called **twin primes**. Mini-Investigation 4.4 lets you search for some interesting prime number patterns, including those involving twin primes, in Table 4.5. To use this table, note that the 43rd prime, for example, appears at the intersection of row 4 and column 3; it is 191.

<i>Make a graph to support your conclusion.</i>	Mini-Investigation 4.4 Finding a Pattern
Do you think that the number of pairs of twin primes increases or decreases with each succeeding interval of 100 prime numbers?	

The table of the first 199 primes raises several interesting questions.

- **Is there a largest prime number?** In answering *no* to this question, Euclid (300 B.C.) gave a clever proof. We present some of the ideas from Euclid's proof here. Euclid claimed that a largest prime number couldn't exist because if there were a largest prime, P , it would be contradicted by the fact that you could always produce a larger prime, N , by multiplying P by all the primes smaller than P and adding 1, namely, $N = (2 \times 3 \times 5 \times 7 \times 11 \times \dots \times P) + 1$. To support this claim, Euclid had to prove two things. The first was that N is larger than P . That wasn't difficult to prove

TABLE 4.5 | The First 199 Prime Numbers

	0	1	2	3	4	5	6	7	8	9
0		2	3	5	7	11	13	17	19	23
1	29	31	37	41	43	47	53	59	61	67
2	71	73	79	83	89	97	101	103	107	109
3	113	127	131	137	139	149	151	157	163	167
4	173	179	181	191	193	197	199	211	223	227
5	229	233	239	241	251	257	263	269	271	277
6	281	283	293	307	311	313	317	331	337	347
7	349	353	359	367	373	379	383	389	397	401
8	409	419	421	431	433	439	443	449	457	461
9	463	467	479	487	491	499	503	509	521	523
10	541	547	557	563	569	571	577	587	593	599
11	601	607	613	617	619	631	641	643	647	653
12	659	661	673	677	683	691	701	709	719	727
13	733	739	743	751	757	761	769	773	787	797
14	809	811	821	823	827	829	839	853	857	859
15	863	877	881	883	887	907	911	919	929	937
16	941	947	953	967	971	977	983	991	997	1009
17	1013	1019	1021	1031	1033	1039	1049	1051	1061	1063
18	1069	1087	1091	1093	1097	1103	1109	1117	1123	1129
19	1151	1153	1163	1171	1181	1187	1193	1201	1213	1217

Estimation Application

A Big Number

Recently, the largest known prime number had 2,098,960 digits.

Estimate the length of a strip of adding machine tape needed to hold this prime if it is typed on one line on the tape. Would it be as long as

- a. less than half a football field?
- b. one football field?
- c. two football fields?
- d. three football fields?
- e. four or more football fields?

Make any necessary assumptions and calculate to check your estimate.

because $1 \times P + 1$ is greater than P , so $(2 \times 3 \times 5 \times 7 \times 11 \times \dots \times P) + 1$ would be greater than $1 \times P + 1$ and hence also greater than P .

The second thing that Euclid had to prove was that $N = (2 \times 3 \times 5 \times 7 \times 11 \times \dots \times P) + 1$ is a prime number. That can also be proven because when N is divided by any prime, the remainder is always 1. A number, N , that isn't divisible by any other prime has only itself and 1 as factors and is, by definition, a prime number. Using this line of reasoning, Euclid was able to convince his mathematician friends, and others, that there is no largest prime number.

- **What is the largest prime number that has been found?** The advent of the computer generated considerable activity in the search for larger and larger prime numbers. In 1978, two high school students, Laura Nickel and Curt Noll, of Hayward, California, showed that $2^{23,209} - 1$ was a prime. As this book went to press, the largest known prime was the number $2^{4,972,593} - 1$. Keeping up with the newest discoveries of largest prime numbers is extremely difficult, and a new largest prime may have been discovered since this book was printed. The estimation application gives a sense of the size of the current largest known prime number mentioned above.
- **Can two primes have an unlimited number of composites between them?** To answer this question, first recall that $n!$, read " n factorial," represents the number $1 \times 2 \times 3 \times 4 \times 5 \times \dots \times n$. For example, $5!$ represents

$1 \times 2 \times 3 \times 4 \times 5$, or 120. Then, if you want to find 100 consecutive composites, you can by using factorials and writing $(101! + 2)$, $(101! + 3)$, $(101! + 4)$, $(101! + 5)$, ..., $(101! + 99)$, $(101! + 100)$, and $(101! + 101)$. The divisibility theorem, $a \mid b$ and $a \mid c \rightarrow a \mid (b + c)$, may be used to show that the first number in the list is divisible by 2, the second by 3, the third by 4, and so on. Therefore each of the 100 consecutive numbers is a composite number. By using any number you choose in place of 101, you can write as many consecutive composites as you want.

- **Are there still some unsolved problems in number theory?** This question can be answered by giving some interesting examples. Christian Goldbach (1690–1764) hypothesized that “every even number greater than 2 is the sum of two primes.” *Goldbach’s conjecture* may be illustrated by noting that $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, $10 = 7 + 3$, and $12 = 7 + 5$. The proof of this conjecture is still one of the unsolved problems in mathematics. The *twin prime conjecture* that “there are an infinite number of pairs of primes whose difference is two” also has not been proven.

Another interesting unsolved problem is proof of the *odd perfect number conjecture* that “there is no odd perfect number.” Although mathematicians have shown that even perfect numbers are all expressible in the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime, the total absence of an odd perfect number has never been verified. Related to this is the unsolved *Mersenne prime conjecture* (Marin Mersenne, 1588–1648) that “there are infinitely many Mersenne primes of the form $2^p - 1$ ” and the lack of verification of the conjecture that there are infinitely many even perfect numbers.

Pierre Fermat (1601–1665), aware of the Pythagorean theorem and the fact that there are numbers a , b , and c such that $a^2 + b^2 = c^2$, conjectured that $a^n + b^n = c^n$ can have no solutions in non-zero integers a , b , and c for $n > 2$. He wrote in the margin of a book, “For this I have discovered a wonderful proof, but the margin is too small to contain it.” Until recently, this *Fermat conjecture* had been an unproved theorem for more than 350 years. However, in October 1994, the English mathematician Andrew Wiles presented a proof of the Fermat conjecture that has been accepted as correct by mathematicians specializing in number theory. His proof is often referred to as “the proof.”

Talk about ways in which you think working on an unsolved number-theory problem might be like the situation(s) you chose from (a) through (d).




Mini-Investigation 4.5 Making a Connection

Which of the following do you think would be like working on an unsolved number-theory problem?

- a. Solving a “whodunnit” mystery
- b. Searching for genealogy connections
- c. Discovering what’s wrong with a car that won’t run
- d. Climbing the tallest mountain in the world

Problems and Exercises | for Section 4.2

A. Reinforcing Concepts and Practicing Skills

1. How many factors does a prime number have?
2. List the first 10 prime numbers.
3. How can you identify a composite number?
4. List the composite numbers between 30 and 40.
5. Which of the following are prime numbers? Why or why not?
 - a. 8 b. 11
 - c. 1 d. 51
 - e. 221
6. Give a counterexample to show that the generalization, all odd numbers are prime, is false.
7. Use factor trees to write the prime factorization of each number.
 - a. 18 b. 48
 - c. 130 d. 51
8. Use the division method to find the prime factorization of each number.
 - a. 504 b. 1176
 - c. 2600 d. 3675
9. Use Table 4.5 on p. 222 to decide how many pairs of twin primes (primes that differ by only 2) there are in prime numbers less than 100.
-  10. Use a calculator to decide if 437 and 541 are prime numbers. Explain how you decided.
-  11. Use a calculator to help you write the prime factorization of 16,731.
12. List factors of the numbers to find the GCF of 42 and 28.
13. Use the intersection of sets method to find the GCF of 12 and 20. Draw a Venn diagram to show the procedure you used.
14. Use the intersection of sets method to find the GCF of 18 and 24. Use set notation to describe the procedure you used.
15. Use a method of your choice to find the GCF of 80 and 124. Explain why you chose to use this method.
16. Use the prime factorization method to find the GCF of each pair of numbers.
 - a. 28, 42 b. 45, 60
 - c. 36, 54
-  17. Use a calculator and the Euclidean algorithm method to find the GCF of each pair of numbers.
 - a. 259, 888 b. 84, 308
 - c. 1232, 7560
18. List the multiples of the numbers to find the LCM of 9 and 12.
19. Use the intersection of sets method to find the LCM of 4 and 18. Draw a Venn diagram to show the procedure you used.
20. Use the intersection of sets method to find the LCM of 6 and 21. Use set notation to show the procedure you used.
21. Use a method of your choice to find the LCM of 18 and 24. Explain why you chose to use this method.
22. Use the prime factorization method to find the LCM of each pair of numbers.
 - a. 27, 36 b. 42, 60
 - c. 28, 40
23. The LCM of a pair of numbers is 36. The product of the numbers is 108. What is the GCF of the numbers?
24. Find the GCF of each triple of numbers.
 - a. 16, 28, 40 b. 30, 36, 48
25. Find the LCM of each triple of numbers.
 - a. 9, 12, 15 b. 28, 40, 56
26. A karate instructor wanted to form small groups from 24 students on Monday and 42 students on Tuesday, with the same number in each of the small groups. What is the largest small-group size possible?
27. What is the shortest length of television cable that could be cut into either a whole number of 18-ft pieces or a whole number of 30-ft pieces?
28. A city siren is set to go off every 24 hours. A civil defense alarm is tested every 36 hours. If the siren and the alarm sound at the same time, after how many hours will they sound at the same time again?
29. A stamp collector has 280 stamps from North America and 264 stamps from South America. The collector wanted to place the same number of stamps on each page of a large stamp book displaying stamps from the Americas. What is the greatest number of stamps the collector can place on each page? How many pages would the book have?
30. Two motorcycles start around a race course at the same time. One cycle passes the starting point every 12 minutes and the other passes it every 15 minutes. How many minutes would elapse before both cycles pass the starting line together?
31. A field is 70 ft by 525 ft. It is to be divided into square garden plots, all the same size, and with sides having whole-number length. What is the largest size garden plot that could be chosen?

B. Deepening Understanding

32. Evaluate the formula $p = n^2 - n + 11$ by substituting each of the numbers 1–11 for n . For which value(s) of n is p not a prime?
33. The formula $p = n^2 - n + 41$ produces primes for $n = 1, 2, 3, \dots, 40$, but $n = 41$ produces a composite number. How can you convince someone that p is composite when $n = 41$ by just looking at the formula?
34. List all the primes less than 100 that are of the form $n^2 + 1$.
35. Marin Mersenne conjectured that, if p is prime, then the number $2^p - 1$ is a prime. Show that this outcome holds for the first four primes. Show that it breaks down for the fifth prime, $p = 11$.
36. Do you think that the formula $p = 6n - 1$ will produce primes more than 50% of the time? Explain your reasoning (*Hint*: Table 4.5 on p. 222 may help.)
37. Finish arranging the numbers 2 through 100 in 6 columns.

2	3	4	5	6
7	8	9	10	11
13	14	15	16	17
...				

Cross out the multiples of 2, 3, 5, and 7. The remaining numbers are prime. Look for and describe patterns.

38. Use Table 4.5 on p. 222 to answer the following questions.
 - a. What can you say about the ones digit of the prime numbers greater than 2?
 - b. What conjecture would you make about the existence and frequency of three consecutive odd number primes?
 - c. Prime numbers such as 13 and 31 are sometimes called *reversal primes*. Find four more pairs of reversal primes.
39. A company executive claimed that his license plate number, 5773, was a prime number. Do you agree? Justify your conclusion.



40. The following BASIC program determines whether an input integer greater than 1 is or is not prime. If the BASIC language is available on your computer, type the program into your computer and try it. Do you think that you could use the program to test any number, no matter how large? Explain.

```

10 INPUT N
20 FOR K = 2 TO SQR(N)
30 IF N/K = INT (N/K) THEN 50
40 GO TO 70
50 PRINT N; " IS NOT PRIME "
    
```

```

60 GO TO 90
70 NEXT K
80 PRINT N; " IS PRIME"
90 PRINT "IF YOU WANT TO CHECK ANOTHER NUM-
    BER, TYPE 1"
100 PRINT "IF NOT, TYPE 0"
110 INPUT C
120 IF C = 1 GO TO 10
130 END
    
```

41. Is the GCF of a pair of numbers always larger or smaller than the LCM of the numbers? Explain with an example.
42. If the GCF of two numbers is 1, what is the LCM of the two numbers? Why?

C. Reasoning and Problem Solving

43. Suppose that E is the set that contains 1 and the even natural numbers, or $E = \{1, 2, 4, 6, 8, 10, 12, 14, \dots\}$. Answer these questions about E .
 - a. Why is 6 a prime in E ?
 - b. What are the first five prime numbers in E ?
 - c. Do you see a pattern for the prime numbers in E ? If so, what is it?
 - d. Do you see a pattern for composite numbers in E ? If so, what is it?
 - e. Does the formula $E = 2e - 2$ generate primes in E ?
 - f. Are there infinitely many primes in E ? Explain.
 - g. What formula generates all composite numbers in E ?
 - h. If e is an element of E , can $2e$ be a prime number in E ? Why?
 - i. Are there any twin primes in E ? (*Hint*: Think about numbers generated from $2e - 2$ and from $2e$, where e is a natural number in E .)
 - j. How does Goldbach's conjecture relate to prime numbers in E ?
44. The following are some interesting theorems about number relationships by the mathematician who conjectured or proved the theorem. Choose some specific numbers and give an example for each theorem.
 - a. Fermat: Every prime number of the form $4x + 1$ is the sum of two unique square numbers.
 - b. Chebyshev: Between every whole number greater than 1 and its double there is at least one prime.
 - c. Euclid: For a natural number p , if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is a perfect number.
 - d. Ulam: If a natural number is even, divide it by 2. If it is odd, multiply it by 3 and add 1. If this process is applied repeatedly, you will arrive at 1. (*Hint*: Try the numbers 16 and 7.)

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45. Give a convincing argument that 2 is the only possible even prime number.
46. Use variables to give a convincing argument that having three consecutive whole numbers that are all prime is impossible.
47. Give a convincing argument that no square number is a prime number.
48. Give two examples to illustrate the possible generalization that $\text{GCF}(m, n) = \text{GCF}(m + n)$, $\text{LCM}(m, n)$. Do the two examples prove that this generalization is true? Explain.
49. When you think about the sums of their proper factors greater than 1, the numbers 48 and 75 have a special relationship. What is this relationship? Show that the numbers 140 and 195 have this same special relationship.
50. Study the following procedure for finding the LCM of two or more numbers.
 - a. The procedure involves starting with the smallest prime and dividing one or the other of the two numbers by that prime as many times as possible before moving to the next prime. The procedure ends when the quotients are both 1. Explain why the procedure works.

```

2 | 90 24
  | 45 12
2 | 45  6
  | 45  3
3 | 45  3
  | 15  1
3 | 15  1
  |  5  1
5 |  5  1
  |  1  1
    
```

$$\text{LCM}(90, 24) = 2^3 \cdot 3^2 \cdot 5.$$


- b. Show how to use the procedure to find the LCM of 24, 28, and 45.
51. Show that this process for finding the number of factors a number has works for the number 24.
 - a. First, express the number as a product of powers of the prime numbers, 2, 3, 5, 7 ...;
 - b. Then, increase each exponent by 1 and calculate the product of the resulting numbers.
52. A student doing a curve-stitching art project knows that the number of different regular polygons he can construct with yarn on a 36-nail board is the number of factors that 36 has, minus 2. How many different polygons can he construct?
53. **The Cheap Watch Problem.** An enterprising watch dealer ordered several identical cheap watches. He kept the cost of each watch secret so that he could sell them to his friends for a lot more than he paid for them. His assistant, checking the invoice, was told that each watch

cost a whole number of dollars, that the cost of a watch was more than the number of watches purchased, and that the total cost was \$437. He quickly used his calculator to figure out how many watches were purchased and what they cost. How did he do it, and what were the number and cost?

54. **The Chewing Gum Box Problem.** A 2-digit number of large boxes contained a prime number of smaller boxes, each of which contained a prime number of packages, each of which contained a prime number of sticks of chewing gum. The total number of sticks of gum was 39,039. If no containers other than the large boxes held more than 15 of the next-smaller objects, and no two containers held the same number of next-size-smaller containers, how many large boxes were there?
55. **The Area Code Problem.** A person has a telephone area code that is a prime number, with no two digits the same. The ones digit is a prime, and when you mark it out, the remaining 2-digit number is also prime. When you mark out the ones and tens digits, the remaining 1-digit number is also a prime. What can you conclude about the state in which this person lives? (*Hint:* Use an area code map from a phone book and Table 4.5 on p. 222.)
56. **Biorhythm Cycles Problem.** Suppose that a baby's biorhythm cycles are together on the day of its birth. The physical cycle is 23 days long, the emotional cycle is 28 days long, and the intellectual cycle is 33 days long. How old will the baby be when the cycles all coincide again?

D. Communicating and Connecting Ideas

57. Make a flow chart that shows a procedure for determining whether a given number is prime.
58. Work with a group of your classmates to explore *lucky numbers*, which are the numbers found by the following process.
 - i. Write down the natural numbers to 100, as in the sieve of Eratosthenes, and mark out numbers as follows.
 - ii. Because 2 is the number following 1 in the list, mark out every second number, leaving the odd numbers.
 - iii. Because 2 has been used as a markout number, 3 is now the first unused number after 1 in the list. Mark out every third number from those remaining. (This step would remove 5, 11, 17, 23, ...)
 - iv. Because 7 is now the first unused number after 1 in the list, mark out every seventh number from those remaining. (This step would remove 19, 39, 61, ...)
 - v. Continue this process, using 9, 13, 15, and 21 as markout numbers. The remaining numbers are the lucky numbers.

- a. How many lucky numbers are there that are less than 100? How many are prime? Composite?
- b. How do the number of twin primes less than 100 compare to the number of twin lucky numbers?
- c. How do the number of prime pairs differing by 4 compare with the number of lucky pairs differing by 4 for the lucky numbers and primes less than 100?
- d. Does Goldbach's conjecture for primes seem to hold true for the lucky numbers?
-  59. If a graphing calculator is available, get help as needed and write a program for utilizing the Euclidean algorithm to find the GCF for any two numbers. (See Appendix A for more information on the graphing calculator.)
60. **Historical Pathways.** The French mathematician Pierre Fermat (1601–1665) is generally considered to be the founder of the theory of numbers. At first, he thought that the formula $P_n = 2^{2^n} + 1$ would always produce primes. Show that it does for $n = 0, 1, 2$, and 3. It also does for $n = 4$, but for $n = 5$ the number produced is 641(6,700,417), which isn't prime.
61. **Making Connections.** Draw a Venn diagram to show how prime numbers, even numbers, odd numbers, composite numbers, and square numbers are related.

CHAPTER SUMMARY

Key Ideas: Questions and Answers

Section 4.1

- **How are the multiples and factors of a number found?** (pp. 189–193) Divide the number by successive whole numbers. If the remainder is 0, the divisor is a factor of the number. You need test only those whole numbers that are no greater than the square root of the number.
- **What is meant by divisibility?** (pp. 193–194) The whole number b is divisible by the number $a \neq 0$ if there is a whole number x so that $ax = b$.
- **Are there some easy ways for deciding whether one number is divisible by another?** (pp. 194–200) The most useful tests are the following. A number is divisible by (a) 2 if its ones digit is 0, 2, 4, 6, or 8; (b) 3 if the sum of its digits is divisible by 3; (c) 4 if the number represented by its last two digits is divisible by 4; (d) 5 if its ones digit is 0 or 5; (e) 6 if it is divisible by both 2 and 3; and (f) 10 if the last digit is 0.
- **How can information about factors be used to classify natural numbers?** (pp. 200–202) Some key classifications are: (a) an even number has 2 as a factor; (b) a square number has an odd number of factors; (c) a perfect number has a sum of its proper factors equal to itself; and (d) certain special numbers have exactly two factors.

Section 4.2

- **What are prime and composite numbers?** (pp. 206–207) A prime number is a natural number that has

exactly two factors. A composite number has more than two factors.

- **What are some ways to find prime numbers?** (pp. 207–212) Use the sieve of Eratosthenes or use a calculator to test for factors.
- **What role do prime numbers play in mathematics?** (pp. 212–214) Every composite number may be expressed as the product of prime numbers in exactly one way, disregarding the order of the factors.
- **How can we find the greatest common factor and least common multiple?** (pp. 214–219) The greatest common factor (GCF) of two numbers is the greatest number that is a factor of both numbers. You can find the GCF by listing factors, by drawing Venn diagrams, by writing prime factorizations, or by using the Euclidean algorithm.

The least common multiple (LCM) of two numbers is the smallest number that is a multiple of both numbers. You can find the LCM by listing factors, by drawing Venn diagrams, or by writing prime factorizations.

- **What are some patterns and relationships involving prime and composite numbers?** (pp. 219–223) Some examples are: (a) There is no largest prime number, but some very large primes have been found; (b) as many consecutive composite numbers as desired may be written; (c) every even number greater than 2 may be the sum of two primes.

Key Terms, Concepts, and Generalizations

Section 4.1

Factor (p. 190)	Divides (p. 194)	Even numbers (p. 200)
Multiple (p. 190)	Divisor (p. 194)	Odd numbers (p. 200)
Factor test theorem (p. 193)	Divisible (p. 194)	Squares (p. 200)
	Divisibility tests (p. 195)	Perfect number (p. 200)

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Deficient number (p. 200)
 Abundant number (p. 200)
 Amicable number (p. 201)

Section 4.2

Prime number (p. 207)
 Composite number (p. 207)

Sieve of Eratosthenes (p. 207)
 Fundamental theorem of arithmetic
 (p. 212)
 Prime factorization (p. 213)
 Greatest common factor (GCF)
 (p. 214)
 Euclidean algorithm (p. 216)

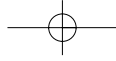
Least common multiple (LCM)
 (p. 217)
 The GCF–LCM theorem (p. 220)
 Relatively prime (p. 220)
 Twin primes (p. 221)
 n factorial (p. 222)

CHAPTER REVIEW**Concepts and Skills**

- List all the factors of each number.
 - 18
 - 32
 - 48
 - 105
- When using a calculator to find the factors of 625, what is the largest number you would have to test?
- Use a calculator to find the factors of each number.
 - 91
 - 143
 - 663
 - 299
- Describe how to decide if a number is divisible by
 - 2.
 - 3.
 - 4.
 - 5.
 - 6.
- Which of the numbers 987, 1436, 4674, and 5580 are divisible by
 - 2?
 - 3?
 - 4?
 - 5?
 - 6?
- Classify as true or false. If false, give a counterexample.
 - If $3 \mid n$, then $9 \mid n$.
 - If $10 \mid n$, then $5 \mid n$.
 - If $2 \mid n$ and $4 \mid n$, then $8 \mid n$.
- How do you know that 7 is a prime number but that 6 is a composite number?
- Which of the following numbers are prime? How do you know?
 - 89
 - 39
 - 137
 - 217
- Write the prime factorization of the numbers 90 and 420 by using
 - a factor tree.
 - the stacked division method.
- Use the prime factorization of 36 to find how many factors 36 has.
- Find the GCF of the numbers 48 and 108 by using
 - the listing factors method.
 - the prime factorization method.
 - the Euclidean algorithm method.
- Find the LCM of the numbers 24 and 32 by using
 - the listing multiples method.
 - the prime factorization method.
- Is each pair of numbers relatively prime? Why or why not?
 - 70, 231
 - 165, 182
- The LCM of a pair of numbers is 72. The product of the numbers is 432. What is the GCF of the numbers? Explain how you know.

Reasoning and Problem Solving

- Give a convincing argument that 2 is the only even prime number.
- You know that a number is divisible by 6 if it is divisible by both 3 and 2. So why isn't a number divisible by 8 if it is divisible by both 4 and 2?
- Produce the smallest number you can that is divisible by 2, 3, 4, 5, 6, and 7. Discuss whether you think a smaller such number exists and why.
- What characteristic do the numbers 8, 10, 15, 26, and 33 have that the numbers 5, 9, 16, 18, and 24 don't have? (*Hint*: List the factors of the numbers.) Give two more numbers that have this characteristic.
- The Synchronized Sirens.** Test siren A was set to go off every 90 minutes. Test siren B was set to go off every 240 minutes. Today both sirens sounded at noon.
 - In how many hours will they both sound at the same time again?
 - How many times will each siren have sounded when they both sound at the same time again after today's noon sounding?
 - How would you set a siren C so that it sounds at different times than A or B but sounds with A and B initially and later they all three sound together again?
- Do you think that the formula $p = 6n + 1$, where n is a whole number, will produce a prime number more than 50% of the time? Give evidence to support your conclusion. (*Hint*: Table 4.5 on page 222 may help.)
- A Corny Experiment.** A corn field 560 meters by 528 meters is to be divided into small square research plots of the same size for experimental purposes.
 - What are the dimensions of the largest research plot that could be chosen?



- b. How many of the largest research plots could be made from the field?
 - c. If you wanted to have more smaller research plots, what other dimension might you use?
- 23. Work with a small group to devise a good way to explain to someone how to find the prime factorization of a number. Give pros and cons of using factor trees, the division method, or a method that you have devised.

Alternative Assessment

- 22. Make a chart showing the different divisibility tests. Explain the connections between some of the tests. Discuss which tests you would encourage students to use and why.
- 24. Analyze the prime factorization methods of finding the GCF and the LCM. Write a description in your own words of how to use these methods. At the end of your descriptions, indicate how these methods are alike and how they are different.

