Doubling Times and the Rule-of-70

The **Rule-of-70** provides a simple way to calculate the approximate number of years it takes for the level of a variable growing at a constant rate to double. This rule states that the approximate number of years n for a variable growing at the constant growth rate of R percent, to double is

$$n = \frac{70}{R}.$$

For example, a city with an annual population growth rate of 5% will double its population in approximately 14 years. If the growth rate were 7%, it would double its population in approximately 10 years.

A more formal analysis of doubling times reveals some interesting characteristics of growth. First, note that finding doubling times requires us to find the set of values of n and r ($= R/100$) such that the ratio of the future level, $X(t + n)$, to the initial level, $X(t)$, equals 2. Referring back to the formula for continuous compounding and dividing each side of that equation by $X(t)$, we find that finding the doubling times involves finding r and n such that

$$e^{r \cdot n} = \frac{X(t + n)}{X(t)} = 2.$$

The first thing to note is that the initial level of the variable does not affect the number of years it takes for the level to double. Increasing both $X(t + n)$ and $X(t)$ does not alter their ratio. Thus the time it takes for a variable to increase by a certain proportion is not a function of its initial level.

What is the special role of the number 70 in this rule? We can answer this question by taking the natural logarithm of each side of the expression $e^{rn} = 2$. This shows that the relationship between r and n that satisfies the doubling time problem is

$$r \cdot n = \ln(2) = 0.6931\ldots.$$

Remembering that the Rule-of-70 is expressed in percentage growth rates, we have $R \cdot n = 69.31$. Thus any product of R and n that equals (approximately) 70 will show us the combination of a percentage growth rate, compounded continuously, and number of years that leads to the doubling of the level of a variable.

Finally, what are the appropriate numbers for trebling times, quadrupling times, and so on? Following the same procedure, we see that the condition for a variable growing continuously at the constant annual rate r to increase by a factor F in n years is

$$e^{r \cdot n} = F.$$

This means that the relationship between the percentage growth rate, R, and the number of years, n, for a variable to increase by a factor F is

$$R \cdot n = 100 \cdot \ln(F).$$

Since $\ln(3) = 1.099$, the rule of thumb for trebling times is called the "rule-of-110." We can figure out the rule of thumb for quadrupling times without a calculator since

$$\ln(4) = \ln(2^2) = 2 \cdot \ln(2).$$

Therefore, the quadrupling time rule is (approximately) the "rule-of-140." If you can remember that $\ln(5) = 1.6$ and $\ln(7) = 1.95$, then you have rules of thumb for any factor between 2 and 10.

In this application we have used the natural logarithm in conjunction with the equation describing continuously compounded growth to analyze the Rule-of-70.