Integration

This notebook will show you how to use *Mathematica* to explore topics introduced in Chapters 5 through 7 of the text.

■ Computing Sums

The *Mathematica* command `Sum[a_j, {j, 1, n}]` is used to compute $\sum_{j=1}^{n} a_j$. For example, the following input instructs *Mathematica* to compute $\sum_{j=1}^{n} (5\, j^2 - 4)$.

```
In[1]:= Sum[5 j^2 – 4, {j, 1, 10}]

Out[1]= 1885
```

The sum can be entered in a more natural way using $\sum_{\square=\square}^{\square} \blacksquare$ in the palette *BasicInput*.

In[2]:=
$$\sum_{j=1}^{10} (5\, j^2 - 4)$$

```
Out[2]= 1885
```

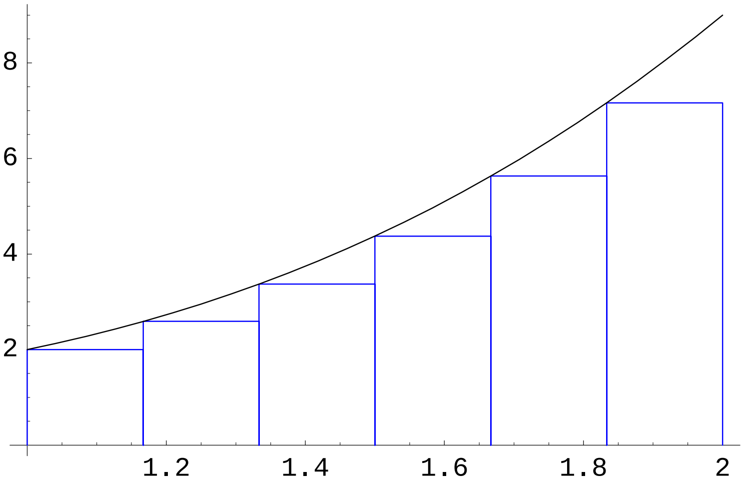
■ Riemann Sums

Since *Mathematica* does not contain a built-in function for illustrating and computing Riemann sums (discussed in Section 5.2 of the text), you can execute the following input cell to create a function which will compute R_n. Don't worry about understanding the code contained here since it involves a couple of advanced concepts which have not been introduced in this manual.

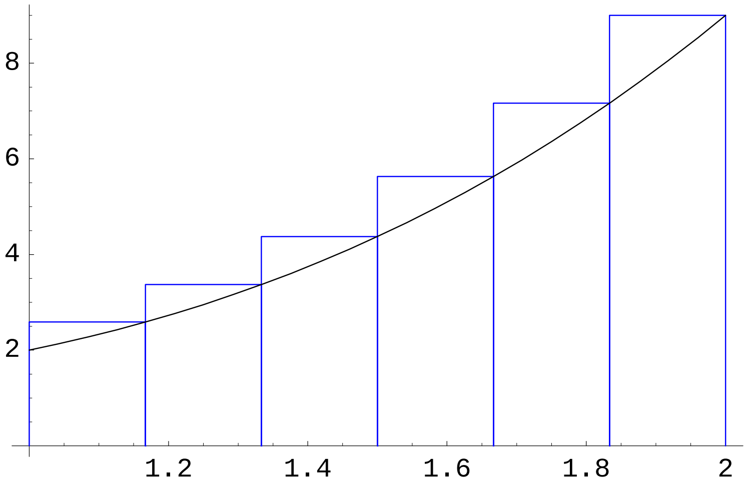
```
                                    b - a
In[3]:= r[a_, b_, n_, m_] := Module[{Δx = -----, recs, est, gr, x, c},
                                      n
          x[i_] := a + i Δx;
          c[i_] := x[i - 1 + m];
                  n
          est =   Σ  f[c[i]] Δx;
                 i=1
          recs = ListPlot[
            Flatten[Table[{
                {x[i - 1], 0}, {x[i - 1], f[c[i]]},
                {x[i], f[c[i]]}, {x[i], 0}}, {i, 1, n}], 1],
            PlotJoined → True, DisplayFunction → Identity,
            PlotRange → All, PlotStyle → RGBColor[0, 0, 1]];
          gr = Plot[f[x], {x, a, b}, DisplayFunction → Identity, PlotRange → All];
          Show[{recs, gr}, DisplayFunction → $DisplayFunction];
          Return[est // N]]
```

To compute a Riemann sum R_n, define the function f which you are using first and then enter `r[a, b, n, 0]` if $c_i = x_{i-1}$, enter `r[a, b, n, 1]` if $c_i = x_i$ or enter `r[a, b, n, .5]` if $c_i = \frac{1}{2}(x_{i-1} + x_i)$. For example, R_6 is computed below for the integral $\int_1^2 (x^3 + 1)\,dx$ using $c_i = x_i$, $c_i = x_{i-1}$ and $c_i = \frac{1}{2}(x_{i-1} + x_i)$, respectively.

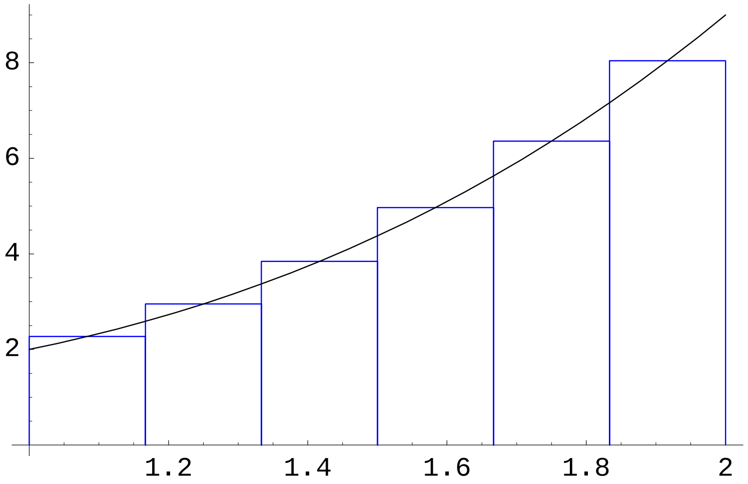
```
In[4]:= f[x_] := x^3 + 1;
        r[1, 2, 6, 0]
        r[1, 2, 6, 1]
        r[1, 2, 6, .5]
```



```
Out[5]= 4.1875
```



```
Out[6]= 5.35417
```



```
Out[7]= 4.73958
```

To compute the Riemann sums without the graphical illustration of the rectangles, first execute the contents of the following input cell.

```
In[8]:= Clear[r];
                                   b - a
        r[a_, b_, n_, m_] := Module[{Δx = -----, recs, est, gr, x, c},
                                     n
          x[i_] := a + i Δx;
          c[i_] := x[i - 1 + m];
                  n
          est =   ∑ f[c[i]] Δx;
                 i=1
          Return[est // N]]
```

In the following cell, R_n is computed for the integral $\int_0^2 [x(x-1)]\,dx$ for values of $n = 10, 20, \ldots, 100$ where $c_i = \frac{1}{2}(x_{i-1} + x_i)$.

```
In[10]:= Clear[f];
         f[x_] = x (x - 1);
         Table[r[0, 2, n, .5], {n, 10, 100, 10}] // TableForm

Out[12]//TableForm=
         0.66
         0.665
         0.665926
         0.66625
         0.6664
         0.666481
         0.666531
         0.666563
         0.666584
         0.6666
```

From the output above, it appears that $\int_0^2 [x(x-1)]\,\mathrm{dx} \approx 0.666$.

■ Indefinite Integrals

Enter `Integrate[f[x], x]` to evaluate $\int f[x]\,ⅆx$.

```
In[13]:= Integrate[3 x^2 - 2 x + 5, x]

Out[13]= 5 x - x^2 + x^3
```

In the above output, note the *Mathematica* does not include a general constant term C and therefore you must remember that $\int (3\,x^2 - 2\,x + 5)\,\mathrm{dx} = x^3 - x^2 + 5\,x + C$.

An indefinite integral can also be evaluated using the button ∫■ ⅆ□ found on the palette *BasicInput*.

```
In[14]:= ∫(3 x^2 - 2 x + 5) ⅆx

Out[14]= 5 x - x^2 + x^3
```

■ Definite Integrals

Enter `Integrate[f[x], {x, a, b}]` to compute $\int_a^b f(x)\,\mathrm{dx}$.

```
In[15]:= Integrate[3 x^2 - 2 x + 5, {x, 1, 2}]

Out[15]= 9
```

On the palette *BasicInput*, click on $\int_{□}^{□}$ ■ ⅆ□ to enter the integral in a more natural form.

```
In[16]:= ∫_1^2 (3 x^2 - 2 x + 5) ⅆx

Out[16]= 9
```

■ Partial Fractions

The `Apart` command will form the standard decomposition of a rational function.

In[17]:= $\mathbf{Apart}\left[\dfrac{1}{(2x+5)(x^2+1)x}\right]$

Out[17]= $\dfrac{1}{5x}-\dfrac{8}{145(5+2x)}+\dfrac{-2-5x}{29(1+x^2)}$

Before looking at the following output, can you guess that value of the integral based upon the partial fractions in the output above?

In[18]:= $\int \% \, dx$

Out[18]= $-\dfrac{2\,\mathrm{ArcTan}[x]}{29}+\dfrac{\mathrm{Log}[x]}{5}-\dfrac{4}{145}\mathrm{Log}[5+2x]-\dfrac{5}{58}\mathrm{Log}[1+x^2]$

Mathematica can be used to compute the integral directly as shown below.

In[19]:= $\int \dfrac{1}{(2x+5)(x^2+1)x}\, dx$

Out[19]= $-\dfrac{2\,\mathrm{ArcTan}[x]}{29}+\dfrac{\mathrm{Log}[x]}{5}-\dfrac{4}{145}\mathrm{Log}[5+2x]-\dfrac{5}{58}\mathrm{Log}[1+x^2]$

■ Simple Differential Equations

Suppose you want to solve the initial value problem, $y' = \frac{1+y}{1+t^2}$, $y(1) = 2$, with the help of *Mathematica*. Separating the variables in the differential equation, you can see that $\frac{1}{1+y}$ dy = $\frac{1}{1+t^2}$ dt. Begin by assigning a name to this differential equation, after integrating both sides.

In[20]:= $\mathbf{eq} = \int \dfrac{1}{1+y}\, dy == \int \dfrac{1}{1+t^2}\, dt + c$

Out[20]= `Log[1 + y] == c + ArcTan[t]`

Now t and y are replaced with the values 1 and 2, , respectively, using the replacement operator `/.` and the `Solve` command is used to find the value of c.

In[21]:= `cvalue = Solve[eq /. {t → 1, y → 2}, c]`

Out[21]= $\left\{\left\{c \to \dfrac{1}{4}(-\pi + 4\,\mathrm{Log}[3])\right\}\right\}$

So the value of c is $\frac{1}{4}(-\pi + 4\ln(3))$ and this value can be recalled by extracting it from the list with `cvalue[[1, 1, 2]]`.

In[22]:= `cvalue[[1, 1, 2]]`

Out[22]= $\dfrac{1}{4}(-\pi + 4\,\mathrm{Log}[3])$

Use the `Solve` command again to find y after replacing c with the value found in the last output.

```
In[23]:= sol = Solve[eq /. c → cvalue[[1, 1, 2]], y]
```

Out[23]= $$\left\{\left\{\texttt{y} \to -1 + e^{\frac{1}{4}\,(-\pi+4\,\texttt{ArcTan[t]}+4\,\texttt{Log[3]})}\right\}\right\}$$

■ Numerical Integration

Suppose you wanted to estimate $\int_0^2 \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ dx using *Mathematica* together with the Trapezoidal rule, Midpoint rule or Simpson's Rule. First, define the function to be integrated.

In[24]:= $$\texttt{f[x\_]} = \frac{1}{\sqrt{2\,\pi}}\, e^{\frac{-x^2}{2}};$$

In this example, $a = 0$, $b = 2$, and for this example, we will let $n = 100$. Then $h = \frac{b-a}{n}$ and we will let the functions `x[i]`, `y[i]`, and `m[i]` represent x_i, y_i, and m_i, respectively. Then using the formulas introduced in Section 5.10 of the text, we can define the variables used in the numerical integration rules.

In[25]:= $$\begin{aligned}&\texttt{Clear[a, b, n, x, y, h];}\\ &\texttt{n = 100; a = 0; b = 2;}\\ &\texttt{h} = \frac{\texttt{b - a}}{\texttt{n}};\\ &\texttt{x[i\_] = 0 + i h;}\\ &\texttt{m[i\_]} = \frac{1}{2}\,\texttt{(x[i - 1] + x[i]);}\\ &\texttt{y[i\_] := f[x[i]];}\end{aligned}$$

To estimate $\int_0^2 \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ dx using the Trapezoidal rule, you need to enter and execute $\frac{h}{2}\,(y[0]\ +\ 2\ \sum_{i=1}^{n-1} y[i] +\ y[n])$. (The command `N[expr, 20]` is also used to round the value of *expr* to 20 digits.)

In[31]:= $$\texttt{N}\left[\frac{\texttt{h}}{2}\left(\texttt{y[0]} + 2\sum_{\texttt{i=1}}^{\texttt{n-1}}\texttt{y[i]} + \texttt{y[n]}\right),\ 20\right]$$

```
Out[31]= 0.47724626867805128894
```

Midpoint Rule:

In[32]:= $$\texttt{N}\left[\texttt{h}\sum_{\texttt{i=1}}^{\texttt{n}}\texttt{f[m[i]]},\ 20\right]$$

```
Out[32]= 0.47725166772970608614
```

Simpson's Rule

In[33]:= $$\begin{aligned}\texttt{N}\Big[\frac{\texttt{h}}{3}\ &\texttt{(y[0] + 4 Sum[y[i], \{i, 1, n - 1, 2\}] +}\\ &\texttt{2 Sum[y[i], \{i, 2, n - 2, 2\}] + y[n]), 20}\Big]\end{aligned}$$

```
Out[33]= 0.47724986795579570810
```

NIntegrate Command

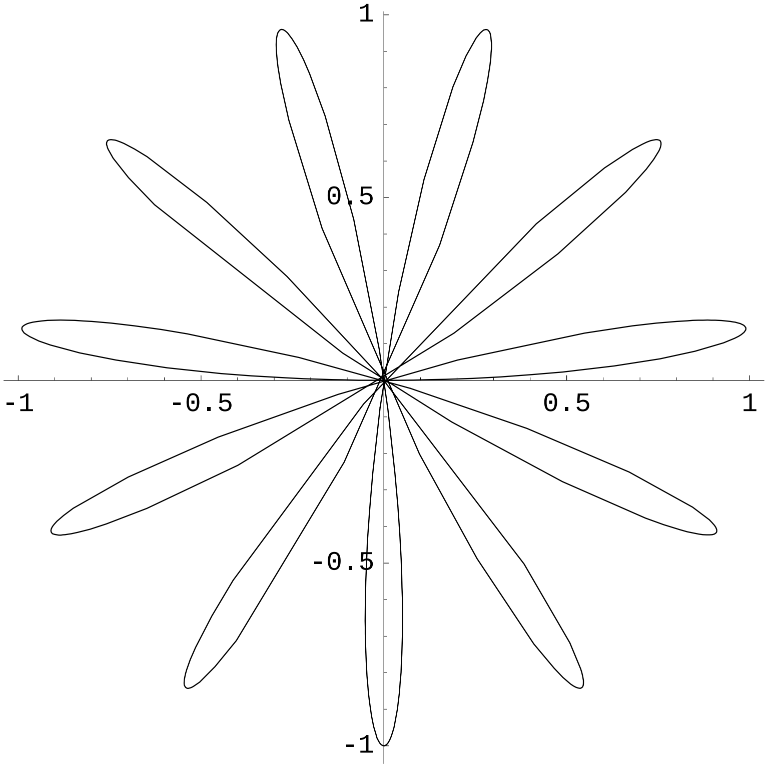
The built-in command, `NIntegrate[f[x], {x, a, b}]` is used to numerically integrate $\int_a^b f(x)\,dx$ using a built-in *Mathematica* algorithm.

```
In[34]:= NIntegrate[f[x], {x, 0, 2}]

Out[34]= 0.47725
```

■ Graphing Polar and Parametric Equations

The Add-on package *Graphics* (in the *Graphics* directory) contains the command `PolarPlot[f[θ], {θ, a, b}]` which will plot the polar equation $r = f(\theta)$ for values of θ from a to b.

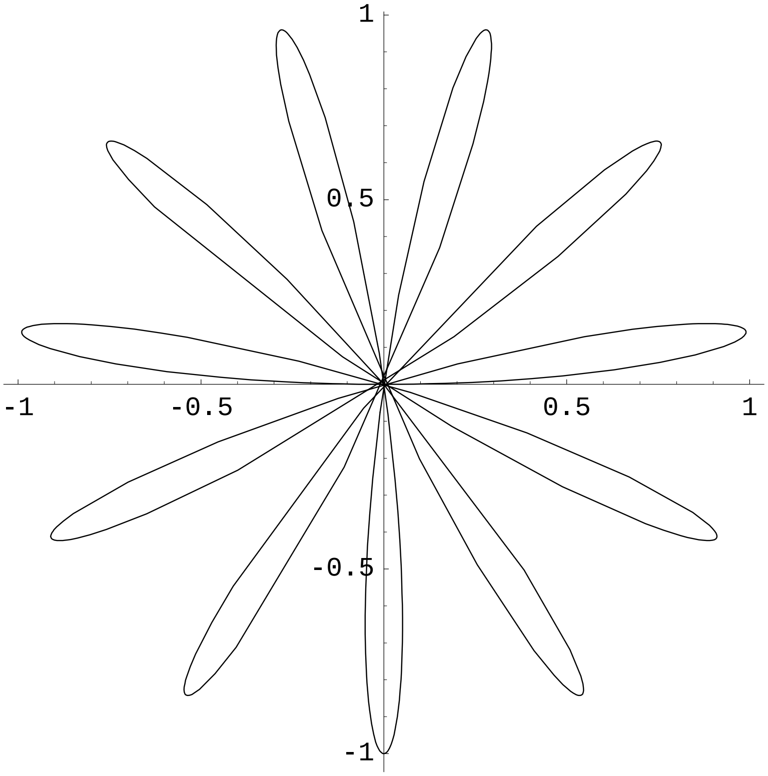
```
In[35]:= << Graphics`Graphics`

In[36]:= PolarPlot[Sin[11 θ], {θ, 0, π}]
```



```
Out[36]= - Graphics -
```

To plot the parametric equation, `g(θ) ⟨Cos[θ], Sin[θ]⟩`, execute `ParametricPlot[g[θ] {Cos[θ], Sin[θ]}, {θ, a, b}]`.

```
In[37]:= gr = ParametricPlot[Sin[11 θ] {Cos[θ], Sin[θ]}, {θ, 0, π}, AspectRatio → 1]
```

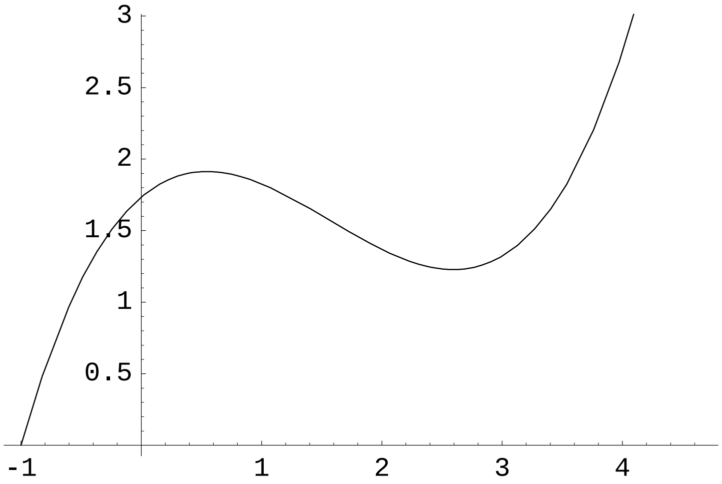


```
Out[37]= - Graphics -
```

■ Computing ArcLength

Suppose you wanted to compute the arclength of the curve below for the given parametric equations.

```
In[38]:= x[t_] = t - Cos[t];
         y[t_] = t + Sin[2 t];
         ParametricPlot[{x[t], y[t]}, {t, 0, 4}]
```



```
Out[40]= - Graphics -
```

Using formula (5) in Section 6.4 of the text, you can attempt to find the value of the arclength with *Mathematica*.

In[41]:= $\int_{0}^{4} \sqrt{\mathtt{x'[t]}^{2} + \mathtt{y'[t]}^{2}}\, \mathrm{d}\mathtt{t}$

Out[41]= $\int_{0}^{4} \sqrt{(1 + 2\,\mathtt{Cos}[2\,\mathtt{t}])^{2} + (1 + \mathtt{Sin}[\mathtt{t}])^{2}}\, \mathrm{d}\mathtt{t}$

Mathematica is unable to find the exact value of the integral, so you can instead use the NIntegrate command instead (or you could use the numerical integration methods introduced in Section 5.10).

In[42]:= $\mathtt{NIntegrate}\left[\sqrt{\mathtt{x'[t]}^{2} + \mathtt{y'[t]}^{2}}, \{\mathtt{t}, 0, 4\}\right]$

Out[42]= `9.33017`

Here is another example where the arclength of the graph of $f(x) = \sin(x)$, from $x = 0$ to $x = 2\pi$, is computed using formula (10) in Section 6.4.

In[43]:= `f[x_] = Sin[x];`

$\mathtt{NIntegrate}\left[\sqrt{1 + \mathtt{f'[x]}^{2}}, \{\mathtt{x}, 0, 2\pi\}\right]$

Out[44]= `7.6404`

■ Improper Integrals

Improper integrals are illustrated in this section using a few simple examples. Suppose you want to compute $\int_{1}^{\infty} \frac{1}{(1+x)^{2}}$ dx using *Mathematica*. By definition, this integral represents $\lim_{b\to\infty} \int_{1}^{b} \frac{1}{(1+x)^{2}}$ dx and therefore the `Limit` command can be used to compute the improper integral.

In[45]:= `Clear[b, x];`

$\mathtt{Limit}\left[\int_{1}^{\mathtt{b}} \frac{1}{(1 + \mathtt{x})^{2}}\, \mathrm{d}\mathtt{x}, \mathtt{b} \to \infty\right]$

Out[46]= $\frac{1}{2}$

Mathematica can also evaluate the improper integral directly if you replace b with ∞.

In[47]:= $\int_{1}^{\infty} \frac{1}{(1 + \mathtt{x})^{2}}\, \mathrm{d}\mathtt{x}$

Out[47]= $\frac{1}{2}$

Since $\int_{1}^{2} \frac{1}{\sqrt{x-1}}\, dx$ converges, *Mathematica* is able to evaluate the integral successfully.

In[48]:= $\int_{1}^{2} \frac{1}{\sqrt{\mathtt{x} - 1}}\, \mathrm{d}\mathtt{x}$

Out[48]= `2`

When you attempt to evaluate the following integral, *Mathematica* outputs a warning message indicating that the integral does not converge.

In[49]:= $$\int_1^2 \frac{1}{(x-1)^{3/2}}\, dx$$

```
Integrate::idiv : Integral of 1/(-1 + x)^(3/2) does not converge on {1, 2}.
```

Out[49]= $$\int_1^2 \frac{1}{(-1+x)^{3/2}}\, dx$$

To determine if the integral diverges to ∞ or $-\infty$ (or neither), use *Mathematica* to compute $\lim_{a\to 1^+} \int_a^2 \frac{1}{(x-1)^{3/2}}\, dx$.

In[50]:=

```
Clear[a];
```

$$\text{Limit}\left[\int_a^2 \frac{1}{(x-1)^{3/2}}\, dx,\ a \to 1,\ \text{Direction} \to -1\right]$$

Out[51]= ∞

■ Taylor Polynomials and Taylor Series

The *Mathematica* command `Normal[Series[f[x], {x, c, n}]]` will find the Taylor polynomial $T_n(x; c)$ for the function $f(x)$. For example, you can use the following command to find T_{11}(x;0) for sin(x).

In[52]:=

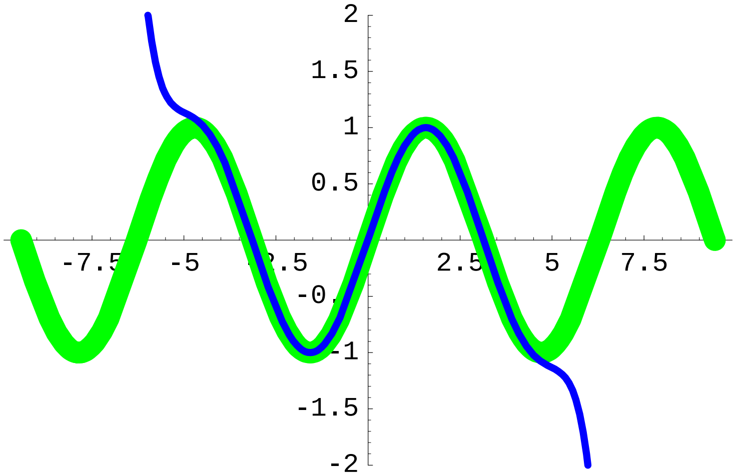
```
Clear[f];
t[x_] = Normal[Series[Sin[x], {x, 0, 11}]]
```

Out[53]= $$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800}$$

By graphing the function together the corresponding Taylor polynomial, you can compare the two functions.

In[54]:=

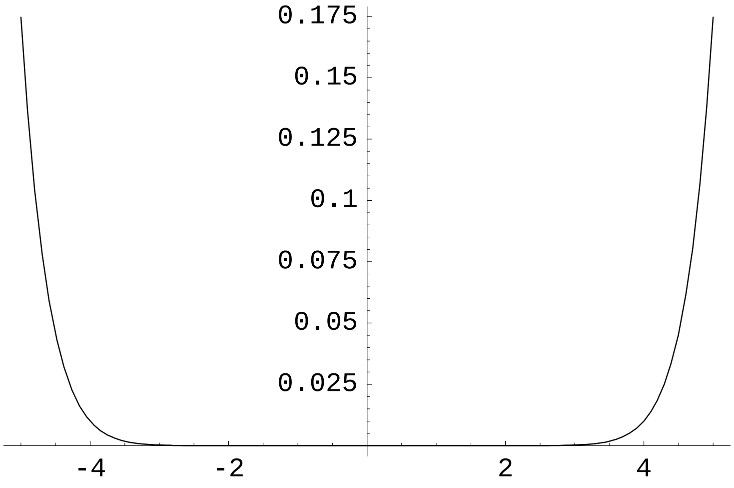
```
Plot[{Sin[x], t[x]}, {x, -3 π, 3 π}, PlotRange → {-2, 2}, PlotStyle →
  {{Thickness[.03], RGBColor[0, 1, 0]}, {Thickness[.01], RGBColor[0, 0, 1]}}]
```



Out[54]= `- Graphics -`

From the graph above, you can see that the polynomial is a "good" approximation of the sine curve from about -5 to 5. To see the difference between the two functions, graph $|\sin(x) - T_{11}(x; 0)|$. From the corresponding graph below, it appear that the difference between sin(x) and $T_{11}(x; 0)$ is less than about 0.175 for $-5 \le x \le 5$.

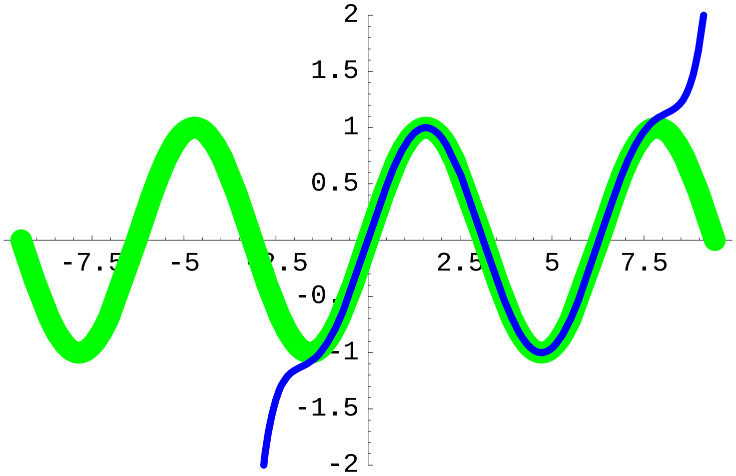
In[55]:=
```
Plot[Abs[Sin[x] - t[x]], {x, -5, 5}, PlotRange → All]
```



Out[55]=
```
- Graphics -
```

Here is another example of a Taylor series using $c = \pi$.

In[56]:=
```
Clear[t];
t[x_] = Normal[Series[Sin[x], {x, π, 11}]];
Plot[{Sin[x], t[x]}, {x, -3 π, 3 π}, PlotRange → {-2, 2}, PlotStyle →
  {{Thickness[.03], RGBColor[0, 1, 0]}, {Thickness[.01], RGBColor[0, 0, 1]}}]
```



Out[58]=
```
- Graphics -
```

The command `Series[f[x], {x, c, n}]` will display the beginning terms of a Taylor series up to powers of *n*. The last term $O[x]^{12}$ represents all the terms with powers of 12 and higher which are not explicitly shown.

In[59]:=
```
Series[Sin[x], {x, 0, 11}]
```

Out[59]=

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800} + O[x]^{12}$$

■ Sequences and Series

The command `Table[a[n], {k, 1, n}]` can be used to generate the first *n* terms of the sequence *a*[1], *a*[2], *a*[3], ... For example, here are the first 50 terms of the sequence $\left\{\frac{\sin(1/n)}{1/n}\right\}_{n=1}^{\infty}$.

In[60]:= $$\mathtt{Table}\left[\frac{\mathtt{Sin}\left[\frac{1}{n}\right]}{\frac{1}{n}}\ \mathtt{//\ N,\ \{n,\ 1,\ 50\}}\right]$$

Out[60]=
```
{0.841471, 0.958851, 0.981584, 0.989616, 0.993347, 0.995377, 0.996602,
 0.997398, 0.997944, 0.998334, 0.998623, 0.998843, 0.999014, 0.99915,
 0.999259, 0.999349, 0.999423, 0.999486, 0.999538, 0.999583, 0.999622,
 0.999656, 0.999685, 0.999711, 0.999733, 0.999753, 0.999771, 0.999787,
 0.999802, 0.999815, 0.999827, 0.999837, 0.999847, 0.999856, 0.999864,
 0.999871, 0.999878, 0.999885, 0.99989, 0.999896, 0.999901, 0.999906, 0.99991,
 0.999914, 0.999918, 0.999921, 0.999925, 0.999928, 0.999931, 0.999933}
```

Now, for example, suppose you wish to compute $\sum_{n=1}^{\infty} \frac{5}{n(n+1)}$. As indicated in Section 7.4 of the text, the value of $\sum_{n=1}^{\infty} \frac{5}{n(n+1)}$ is the limit of the sequence $\left\{\sum_{n=1}^{k} \frac{5}{n(n+1)}\right\}_{k=1}^{\infty}$. The output below shows the first 200 terms of this sequence.

In[61]:= $$\mathtt{Table}\left[\sum_{\mathtt{n=1}}^{\mathtt{k}}\left(\frac{\mathtt{5}}{\mathtt{n\ (n+1)}}\right)\ \mathtt{//\ N,\ \{k,\ 1,\ 200\}}\right]$$

Out[61]=
```
{2.5, 3.33333, 3.75, 4., 4.16667, 4.28571, 4.375, 4.44444, 4.5, 4.54545,
 4.58333, 4.61538, 4.64286, 4.66667, 4.6875, 4.70588, 4.72222, 4.73684,
 4.75, 4.7619, 4.77273, 4.78261, 4.79167, 4.8, 4.80769, 4.81481,
 4.82143, 4.82759, 4.83333, 4.83871, 4.84375, 4.84848, 4.85294, 4.85714,
 4.86111, 4.86486, 4.86842, 4.87179, 4.875, 4.87805, 4.88095, 4.88372,
 4.88636, 4.88889, 4.8913, 4.89362, 4.89583, 4.89796, 4.9, 4.90196,
 4.90385, 4.90566, 4.90741, 4.90909, 4.91071, 4.91228, 4.91379, 4.91525,
 4.91667, 4.91803, 4.91935, 4.92063, 4.92188, 4.92308, 4.92424, 4.92537,
 4.92647, 4.92754, 4.92857, 4.92958, 4.93056, 4.93151, 4.93243, 4.93333,
 4.93421, 4.93506, 4.9359, 4.93671, 4.9375, 4.93827, 4.93902, 4.93976,
 4.94048, 4.94118, 4.94186, 4.94253, 4.94318, 4.94382, 4.94444, 4.94505,
 4.94565, 4.94624, 4.94681, 4.94737, 4.94792, 4.94845, 4.94898, 4.94949,
 4.95, 4.9505, 4.95098, 4.95146, 4.95192, 4.95238, 4.95283, 4.95327,
 4.9537, 4.95413, 4.95455, 4.95495, 4.95536, 4.95575, 4.95614, 4.95652,
 4.9569, 4.95726, 4.95763, 4.95798, 4.95833, 4.95868, 4.95902, 4.95935,
 4.95968, 4.96, 4.96032, 4.96063, 4.96094, 4.96124, 4.96154, 4.96183,
 4.96212, 4.96241, 4.96269, 4.96296, 4.96324, 4.9635, 4.96377, 4.96403,
 4.96429, 4.96454, 4.96479, 4.96503, 4.96528, 4.96552, 4.96575, 4.96599,
 4.96622, 4.96644, 4.96667, 4.96689, 4.96711, 4.96732, 4.96753, 4.96774,
 4.96795, 4.96815, 4.96835, 4.96855, 4.96875, 4.96894, 4.96914,
 4.96933, 4.96951, 4.9697, 4.96988, 4.97006, 4.97024, 4.97041, 4.97059,
 4.97076, 4.97093, 4.9711, 4.97126, 4.97143, 4.97159, 4.97175, 4.97191,
 4.97207, 4.97222, 4.97238, 4.97253, 4.97268, 4.97283, 4.97297,
 4.97312, 4.97326, 4.9734, 4.97354, 4.97368, 4.97382, 4.97396, 4.97409,
 4.97423, 4.97436, 4.97449, 4.97462, 4.97475, 4.97487, 4.975, 4.97512}
```

From the output above, the sequence of partial sums appears to converge to 5. To verify this conjecture, the limit is computed to obtain $\sum_{n=1}^{\infty}\left(\frac{5}{n(n+1)}\right)$.

In[62]:= $\mathtt{Limit}\left[\sum_{n=1}^{k}\left(\frac{5}{n\,(n+1)}\right),\ k\to\infty\right]$ `// Simplify`

Out[62]= 5

Mathematica can also compute the limit directly.

In[63]:= $\sum_{n=1}^{\infty}\left(\frac{5}{n\,(n+1)}\right)$

Out[63]= 5