Chapter 2 Finding the Derivative

Section 2.1 Derivatives of Polynomials

1. $f(x) = 4x - 7$;

$$\begin{aligned} f'(x) &= (4x-7)' \\ &= (4x)' - (7)' \\ &= 4(x)' - (7)' \\ &= 4(1) - 0 \\ &= 4 \end{aligned}$$

3. $y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1$;

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1\right)' \\ &= \left(\frac{1}{4}x^4\right)' + \left(\frac{1}{3}x^3\right)' + \left(\frac{1}{2}x^2\right)' + (x)' + (1)' \\ &= \frac{1}{4}(x^4)' + \frac{1}{3}(x^3)' + \frac{1}{2}(x^2)' + (x)' + (1)' \\ &= \frac{1}{4}(4x^3) + \frac{1}{3}(3x^2) + \frac{1}{2}(2x) + 1 + 0 \\ &= x^3 + x^2 + x + 1 \end{aligned}$$

5. $H(s) = \pi^2$;

$H'(s) = (\pi^2)' = 0$

7. $f(x) = (x+1)(2x+1) = 2x^2 + 3x + 1$;

$$\begin{aligned} f'(x) &= (2x^2 + 3x + 1)' \\ &= (2x^2)' + (3x)' + (1)' \\ &= 2(x^2)' + 3(x)' + (1)' \\ &= 2(2x) + 3(1) + 0 \\ &= 4x + 3 \end{aligned}$$

9. **a.** If $f(x) = \frac{1}{3}x^3$, then

$$\begin{aligned} f'(x) &= \left(\frac{1}{3}x^3\right)' \\ &= \frac{1}{3}(x^3)' \\ &= \frac{1}{3}(3x^2) \\ &= x^2. \end{aligned}$$

b. We seek a function $g(x)$ with $g'(x) = f'(x)$. Since the derivative of a constant function is 0 and since the derivative of a sum is the sum of the derivatives, we consider the function $g(x) = f(x) + 10 = \frac{1}{3}x^3 + 10$.

Then

$$\begin{aligned} g'(x) &= \left(\frac{1}{3}x^3 + 10\right)' \\ &= \left(\frac{1}{3}x^3\right)' + (10)' \\ &= x^2 + 0 \\ &= x^2. \end{aligned}$$

We note that any function of the form $g(x) + C$ where C is any constant is a function with the same derivative as $g(x)$.

11. **a.** If $y = x^4$, then $\frac{dy}{dx} = (x^4)' = 4x^3$.

b. We seek a function $h(x)$ with derivative $\frac{1}{3}x^3$. Since $(x^4)' = 4x^3$ and since the derivative of a constant times a function is the constant times the derivative of a function, we know that $(Cx^4)' = 4Cx^3$. So, find C such that $4C = \frac{1}{3}$. We conclude that $C = \frac{1}{12}$ and, hence, $\left(\frac{1}{12}x^4\right)' = \frac{1}{3}x^3$ so $h(x) = \frac{1}{12}x^4$.

We note that any function of the form $f(x) + C$ where C is any constant is a function with the same derivative as $f(x)$.

In problems 13–15 we have functions *f*, *g*, and *h* with $f'(3) = -4$, $g'(3) = 7$, and $h'(3) = 1$.

13. If $s(x) = 3f(x) - 2g(x) + \frac{1}{2}h(x)$, then

$$\begin{aligned}
s'(3) &= \left.\left(3f(x) - 2g(x) + \frac{1}{2}h(x)\right)'\right|_{x=3} \\
&= \left.\left((3f(x))' - (2g(x))' + \left(\frac{1}{2}h(x)\right)'\right)\right|_{x=3} \\
&= \left.\left(3f'(x) - 2g'(x) + \frac{1}{2}h'(x)\right)\right|_{x=3} \\
&= 3f'(3) - 2g'(3) + \frac{1}{2}h'(3) \\
&= 3(-4) - 2(7) + \frac{1}{2}(1) \\
&= \frac{-51}{2}.
\end{aligned}$$

15. We first solve $f(x) + g(x) + h(x) + s(x) = 20\sqrt{3}$ for $s(x)$ obtaining
$s(x) = 20\sqrt{3} - (f(x) + g(x) + h(x))$.
The derivative of $s(x)$ when $x = 3$ is given by

$$\begin{aligned}
s'(x) &= \left.\left(20\sqrt{3} - (f(x) + g(x) + h(x))\right)'\right|_{x=3} \\
&= \left.\left(20\sqrt{3}\right)' - (f(x))' - (g(x))' - (h(x))'\right|_{x=3} \\
&= \left(20\sqrt{3}\right)' - f'(3) - g'(3) - h'(3) \\
&= 0 - (-4) - (7) - (1) \\
&= -4.
\end{aligned}$$

17. We seek the equation of the tangent line to the graph of $f(x) = 2x^2 - 3x + 4$ at the point (2, 6). We first compute $f'(2)$:

$$\begin{aligned}
f'(2) &= \left.(2x^2 - 3x + 4)'\right|_{x=2} \\
&= \left.\left((2x^2)' - (3x)' + (4)'\right)\right|_{x=2} \\
&= \left.\left(2(x^2)' - 3(x)' + (4)'\right)\right|_{x=2} \\
&= \left.(2(2x) - 3(1) + 0)\right|_{x=2} \\
&= \left.(4x - 3)\right|_{x=2} \\
&= 5.
\end{aligned}$$

The tangent line to the curve is given by $y - 6 = f'(2)(x - 2)$ or $y - 6 = 5(x - 2)$.

19. We seek the equation of the tangent line to the graph of $g(x) = ax^2 + bx + c$ at the point $(d, g(d))$. We first compute $g'(x)$: $g'(x) = 2ax + b$. At $x = d$ we have $g'(d) = 2ad + b$ and $g(d) = ad^2 + bd + c$.
The tangent line to the curve is given by
$y - g(d) = g'(d)(x - d)$ or
$y - (ad^2 + bd + c) = (2ad + b)(x - d)$.

21. Multiply the terms on the right in each and simplify.

23. By the definition of a derivative we have

$$\begin{aligned}
(cf)'(x) &= \lim_{z \to x} \frac{cf(z) - cf(x)}{z - x} \\
&= \lim_{z \to x} c\left(\frac{f(z) - f(x)}{z - x}\right) \\
&= c \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\
&= cf'(x).
\end{aligned}$$

25. By the definition of the derivative and the difference-of-*n*th-powers factorization we have

$$\begin{aligned}
\left(\frac{1}{x^n}\right)' &= \lim_{z \to x} \frac{\frac{1}{z^n} - \frac{1}{x^n}}{z - x} \\
&= \lim_{z \to x} \frac{x^n - z^n}{x^n z^n (z - x)} \\
&= \lim_{z \to x} \frac{-(z - x)(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + zx^{n-2} + x^{n-1})}{x^n z^n (z - x)} \\
&= \lim_{z \to x} \frac{-(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + zx^{n-2} + x^{n-1})}{x^n z^n} \\
&= \frac{-nx^{n-1}}{x^{2n}} \\
&= \frac{-n}{x^{n+1}} \\
&= -nx^{-n-1}.
\end{aligned}$$

27. If $r(\theta) = \theta + 6 + \frac{1}{\theta}$, then

$$\begin{aligned} r'(\theta) &= \left(\theta + 6 + \frac{1}{\theta}\right)' \\ &= \theta' + 6' + \left(\frac{1}{\theta}\right)' \\ &= 1 + \frac{-1}{\theta^2}. \end{aligned}$$

29. If $h(r) = \left(1 + \frac{2}{r^2}\right)^2 = 1 + \frac{4}{r^2} + \frac{4}{r^4}$, then

$$\begin{aligned} h'(r) &= \left(1 + \frac{4}{r^2} + \frac{4}{r^4}\right)' \\ &= (1)' + \left(\frac{4}{r^2}\right)' + \left(\frac{4}{r^4}\right)' \\ &= (1)' + 4\left(\frac{1}{r^2}\right)' + 4\left(\frac{1}{r^4}\right)' \\ &= 0 + 4\frac{-2}{r^3} + 4\frac{-4}{r^5} \\ &= \frac{-8}{r^3} + \frac{-16}{r^5}. \end{aligned}$$

31. If Matt is climbing the rope at twice the rate of Bart and Bart climbs $2000 - 10 = 1990$ feet in 20 minutes, then Matt must climb 2(190) = 380 feet in that same 20 minute period. Since Bart started at ground level, after 20 minutes we conclude that he climbed 380 feet.
If $B(t)$ is Bart's height after t minutes and $M(t)$ is Matt's height after t minutes then $B'(t)$ is the rate at which Bart is climbing and $M'(t)$ is the rate at which Matt is climbing. Since Matt is ascending twice as fast as Bart, $M'(t) = 2B'(t)$.

Section 2.2 Derivatives of Products and Quotients

1. $y=(x^2-3x+1)(2x+4)$;

$$\begin{aligned} y' &= (x^2-3x+1)'(2x+4)+(x^2-3x+1)(2x+4)' \\ &= (2x-3)(2x+4)+(x^2-3x+1)(2) \\ &= 6x^2-4x-10 \end{aligned}$$

3. $H(t)=(-9t^5+6t^4-21t^2+8t-1)^2$;

Set $h(t)=(-9t^5+6t^4-21t^2+8t-1)$. Then

$$\begin{aligned} H'(t) &= (h(t)^2)' \\ &= (h(t)h(t))' \\ &= h'(t)h(t)+h(t)h'(t) \\ &= 2h(t)h'(t) \end{aligned}$$

It follows that $H'(t)=2(-9t^5+6t^4-21t^2+8t-1)(-45t^4+24t^3-42t+8)$.

5. $f(x)=\dfrac{2x^2-3x+10}{x^2-7}$;

$$\begin{aligned} f'(x) &= \frac{(2x^2-3x+10)'(x^2-7)-(2x^2-3x+10)(x^2-7)'}{(x^2-7)^2} \\ &= \frac{(4x-3)(x^2-7)-(2x^2-3x+10)(2x)}{(x^2-7)^2} \\ &= \frac{3(7-16x+x^2)}{(x^2-7)^2} \end{aligned}$$

7. $s(r)=\left(1-\dfrac{1}{r}+\dfrac{2}{r^2}\right)^2$;

In view of the solution to Exercise 3,

$s'(r)=2\left(1-\dfrac{1}{r}+\dfrac{2}{r^2}\right)\left(1-\dfrac{1}{r}+\dfrac{2}{r^2}\right)'$. Thus,

$$\begin{aligned} s'(r) &= 2\left(1-\frac{1}{r}+\frac{2}{r^2}\right)\left((1)'-\left(\frac{1}{r}\right)'+\left(\frac{2}{r^2}\right)'\right) \\ &= 2\left(1-\frac{1}{r}+\frac{2}{r^2}\right)\left(\frac{1}{r^2}-\frac{4}{r^3}\right) \end{aligned}$$

9. $r=(\theta^2-2)(\theta^2-3)(\theta^2-4)$;

$$\begin{aligned} \frac{dr}{d\theta} &= [\{(\theta^2-2)(\theta^3-3)\}(\theta^4-4)]' \\ &= \{(\theta^2-2)(\theta^3-3)\}'(\theta^4-4)+\{(\theta^2-2)(\theta^3-3)\}(\theta^4-4)' \\ &= \{(\theta^2-2)'(\theta^3-3)+(\theta^2-2)(\theta^3-3)'\}(\theta^4-4)+\{(\theta^2-2)(\theta^3-3)\}(\theta^4-4)' \\ &= \{2\theta(\theta^3-3)+(\theta^2-2)(3\theta^2)\}(\theta^4-4)+\{(\theta^2-2)(\theta^3-3)\}(4\theta^3) \\ &= 2\theta(\theta^3-3)(\theta^4-4)+(\theta^2-2)(3\theta^2)(\theta^4-4)+(\theta^2-2)(\theta^3-3)(4\theta^3) \end{aligned}$$

11. $p(r)=\sqrt{r^2+2r+7}$. Note that the radicand is positive for all values of r. By Example 2, we have

$$\begin{aligned}p'(r)&=\left(\sqrt{r^2+2r+7}\right)'\\&=\frac{1}{2}\frac{(r^2+2r+7)'}{\sqrt{r^2+2r+7}}\\&=\frac{1}{2}\frac{(2r+2)}{\sqrt{r^2+2r+7}}\\&=\frac{r+1}{\sqrt{r^2+2r+7}}.\end{aligned}$$

13. $F(s)=\sqrt{2s-1}\sqrt{6s+7}$. Note that the domain for the function F is $\left[\frac{1}{2},\ \infty\right)$. Using the product rule and Example 2, we have

$$\begin{aligned}&F'(s)\\&=\left(\sqrt{2s-1}\right)'\sqrt{6s+7}+\sqrt{2s-1}\left(\sqrt{6s+7}\right)'\\&=\frac{1}{2}\frac{(2s-1)'}{\sqrt{2s-1}}\sqrt{6s+7}+\sqrt{2s-1}\left(\frac{1}{2}\frac{(6s+7)'}{\sqrt{6s+7}}\right)\\&=\frac{1}{2}\frac{2}{\sqrt{2s-1}}\sqrt{6s+7}+\sqrt{2s-1}\left(\frac{1}{2}\frac{6}{\sqrt{6s+7}}\right).\end{aligned}$$

In problems 15–17 we have functions u, v, and w with $u(2)=1$, $v(2)=2$, $w(2)=3$, $u'(2)=4$, $v'(2)=5$, $w'(2)=6$.

15. We seek $f'(2)$ given that $f(x)=u(x)v(x)$. By the product rule, $f(x)=u'(x)v(x)+u(x)v'(x)$. Hence,

$$f'(2)=u'(2)v(2)+u(2)v'(2)=4(2)+1(5)=13.$$

17. Find $f'(2)$ given that $f(x)=\sqrt{2w(x)-3u(x)}$. By Example 2, we have

$$\begin{aligned}f'(x)&=\left(\sqrt{2w(x)-3u(x)}\right)'\\&=\frac{1}{2}\frac{(2w(x)-3u(x))'}{\sqrt{2w(x)-3u(x)}}\\&=\frac{1}{2}\frac{2w'(x)-3u'(x)}{\sqrt{2w(x)-3u(x)}}\end{aligned}$$

Now, we find $f'(2)$:

$f'(2)=\frac{1}{2}\frac{2(6)-3(4)}{\sqrt{2(3)-3(1)}}=0$. (It is implicitly assumed that the function $2w(x)-3u(x)>0$ at and near 2.)

19. If $y=(2x^2-1)(3x+7)$, find all $(x,\ y)$ so that $\frac{dy}{dx}=0$. We first compute $\frac{dy}{dx}$, then we solve the equation $\frac{dy}{dx}=0$ for x, and then determine the y-values corresponding to the points where $\frac{dy}{dx}=0$.

$$\begin{aligned}\frac{dy}{dx}&=(2x^2-1)'(3x+7)+(2x^2-1)(3x+7)'\\&=(4x)(3x+7)+(2x^2-1)(3)\\&=18x^2+28x-3\end{aligned}$$

Now, $\frac{dy}{dx}=0$ if and only if $-3+28x+18x^2=0$. Applying the quadratic formula, we find that $x=\frac{-14\pm 5\sqrt{10}}{18}$. Hence, the desired points are

$$\left(\frac{-14-5\sqrt{10}}{18},\ \frac{238+625\sqrt{10}}{243}\right)$$
$$\approx(-1.65619,\ 9.11285)$$

and

$$\left(\frac{-14+5\sqrt{10}}{18},\ \frac{238-625\sqrt{10}}{243}\right)$$
$$\approx(0.100633,\ -7.15401).$$

21. The equation of the tangent line to $y=\frac{2x^2-7}{7x+1}$ at $x=2$ is given by $y-y(2)=y'(2)(x-2)$. Substituting $x=2$ into the given function relation yields $y(2)=\frac{2(2)^2-7}{7(2)+1}=\frac{1}{15}$. We next determine $\frac{dy}{dx}$:

$$\begin{aligned}\frac{dy}{dx}&=\left(\frac{2x^2-7}{7x+1}\right)'\\&=\frac{(2x^2-7)'(7x+1)-(2x^2-7)(7x+1)'}{(7x+1)^2}\\&=\frac{4x(7x+1)-(2x^2-7)7}{(7x+1)^2}\\&=\frac{49+4x+14x^2}{(7x+1)^2}\end{aligned}$$

Evaluating $\frac{dy}{dx}$ at $x=2$ produces the value of $\frac{113}{225}$. So, the equation for the tangent line is $y-\frac{1}{15}=\frac{113}{225}(x-2)$.

23. We seek functions $u(x)$ and $v(x)$ so that if $w(x) = u(x)\ v(x)$,

$$\begin{aligned} w'(x) &= (2x-3)(4x^3+3x-1) \\ &\quad + (x^2-3x+8)(12x^2+3). \end{aligned}$$

Apply the product rule to the function $w(x)$ gives $w'(x) = u'(x)v(x) + u(x)v'(x)$. We conclude that $u(x) = x^2 - 3x + 8$ and $v(x) = 4x^3 + 3x - 1$.

25. Assume $u(x)$ is differentiable.

a. By the product rule,

$$\begin{aligned} (u(x)^2)' &= (u(x)u(x))' \\ &= u'(x)u(x) + u(x)u'(x) \\ &= 2u(x)u'(x). \end{aligned}$$

b. By part **a** and the product rule,

$$\begin{aligned} (u(x)^3)' &= (u(x)[u(x)^2])' \\ &= u'(x)[u(x)^2] + u(x)[u(x)^2]' \\ &= u'(x)u(x)^2 + u(x)[2u(x)u'(x)] \\ &= 3u(x)^2u'(x). \end{aligned}$$

c. By part **b** and the product rule,

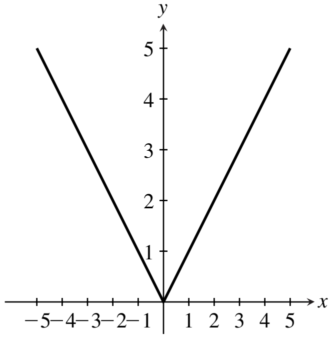
$$\begin{aligned} (u(x)^4)' &= (u(x)[u(x)^3])' \\ &= u'(x)[u(x)^3] + u(x)[u(x)^3]' \\ &= u'(x)u(x)^3 + u(x)[3u(x)^2u'(x)] \\ &= 4u(x)^3u'(x). \end{aligned}$$

d. By part **c** and the product rule,

$$\begin{aligned} (u(x)^5)' &= (u(x)[u(x)^4])' \\ &= u'(x)[u(x)^4] + u(x)[u(x)^3]' \\ &= u'(x)u(x)^4 + u(x)[4u(x)^3u'(x)] \\ &= 5u(x)^4u'(x). \end{aligned}$$

Based on the solutions to parts **a**–**d**, we conjecture that $(u(x)^n)' = nu(x)^{n-1}u'(x)$.

27. $\left(|x|\right)' = \left(\sqrt{x^2}\right)' = \dfrac{1}{2}\dfrac{(x^2)'}{\sqrt{x^2}} = \dfrac{1}{2}\dfrac{2x}{|x|} = \dfrac{x}{|x|}$. The formula for $\left(|x|\right)'$ suggests that the derivative is defined for all real numbers except 0. The graph of $y = |x|$ shown has a corner at the point (0, 0) and, so, $\left(|x|\right)'$ is defined away from zero.



29. All references are with respect to Figure 2.4. If one pulls at P so that wheel A turns one complete revolution clockwise, then the chain is moved through a distance of $2\pi R$ units in the direction of b via the larger radius pulley on wheel A. However, at the same time the smaller radius pulley is allowing the chain to pass over it in the direction of A at the rate of $2\pi r$ units per complete revolution. Hence, the gross change in chain length is $2\pi R - 2\pi r$. Since the gross change in chain length is divided over the "up" and "down" sides of the chain, the net change is $\pi(R - r)$.

31. Since $M(t) = D(t)V(t)$

$$M'(t) = D'(t)V(t) + D(t)V'(t).$$

So, when $D(t) = 1200\ \text{kg/m}^3$, $V(t) = 0.01\ \text{m}^3$ $D'(t) = 0.001\ \text{kg/m}^3/\text{minute}$ and $V'(t) = 0.0005\ \text{m}^3/\text{minute}$ we get

$$\begin{aligned} M'(t) &= (0.001)(0.01) \\ &\quad + (1200)(0.0005)\ \text{kg/minute} \\ &= 0.60001\ \text{kg/minute} \end{aligned}$$

33. The area of the circular oil slick is given by the equation $A = \pi r^2$ and the volume is determined by $V = \pi r^2 h$. We note that all the variables (r, A, V, h) depend on time. Since the radius is increasing at the rate of 10 meters per hour, we know that $\frac{dr}{dt} = 10$ m/h. So, differentiating both sides of $A = \pi r^2$ with respect to time yields

$$\frac{dA}{dt} = \pi(r^2)' = \pi\left(\frac{dr}{dt} r + r\frac{dr}{dt}\right) = 2\pi r \frac{dr}{dt}.$$

At the instant that the radius is 2 km (= 2000 m), we have that

$$\begin{aligned} \left.\frac{dA}{dt}\right|_{r=2000} &= 2\pi(2000)(10) \text{ m}^2/\text{h} \\ &= 40{,}000\pi \text{ m}^2/\text{h} \end{aligned}$$

Now, differentiating both sides of $V = \pi r^2 h$ produces

$$\begin{aligned} \frac{dV}{dt} &= (\pi r^2 h)' \\ &= \pi((r^2)'h + r^2 h') \\ &= \pi((r'r + rr')h + r^2 h') \\ &= \pi(2rr'h + r^2 h') \end{aligned}$$

Since $h = 0.005$ meters, $h' = 0$. So, at the moment $r = 2000$ meters we have that

$$\frac{dV}{dt} = \pi 2(2000)(10)(0.005) = 200\pi \text{ m}^3/\text{h}$$

35. We derive the quotient rule for derivatives. We assume that $u(x)$ and $v(x)$ are differentiable at all points in the domain of $w(x) = \frac{u(x)}{v(x)}$. Further, we assume that $v(x) \neq 0$.

We apply the definition of a derivative to the function $w(x)$:

$$\begin{aligned} & w'(x) \\ &= \lim_{t\to x} \frac{\frac{u(t)}{v(t)} - \frac{u(x)}{v(x)}}{t - x} \\ &= \lim_{t\to x} \frac{u(t)v(x) - u(x)v(t)}{t - x} \frac{1}{v(t)v(x)} \\ &= \lim_{t\to x} \frac{u(t)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(t)}{(t - x)v(t)v(x)} \\ & \quad (\text{Note: } -u(x)v(x) + u(x)v(x) = 0) \\ &= \lim_{t\to x} \frac{(u(t) - u(x))v(x) + (-u(x))(v(t) - v(x))}{(t - x)v(t)v(x)} \\ &= \lim_{t\to x} \frac{\frac{u(t)-u(x)}{t-x} v(x) - u(x) \frac{v(t)-v(x)}{t-x}}{v(t)v(x)} \\ &= \frac{\left(\lim_{t\to x} \frac{u(t)-u(x)}{t-x}\right) v(x) - u(x) \left(\lim_{t\to x} \frac{v(t)-v(x)}{t-x}\right)}{\left(\lim_{t\to x} v(t)\right) v(x)} \end{aligned}$$

By the definition of the derivative

$$\lim_{t\to x} \frac{u(t) - u(x)}{t - x} = u'(x)$$

and $\lim_{t\to x} \frac{v(t) - v(x)}{t - x} = v'(x)$.

Further, because v has a derivative at x, it is continuous at x. Thus $\lim_{t\to x} v(t) = v(x)$.

Substitution into the last expression for $w'(x)$ returns the quotient rule.

37. We know that $(uv)'(a)=\lim\limits_{t\to a}\dfrac{u(t)v(t)-u(a)v(a)}{t-a}$

Substituting for $u(t)$ and $v(t)$ we get

$$\lim_{t\to a}\frac{(u(a)+u'(a)(t-a)+E_u(t)(t-a))(v(a)+v'(a)(t-a)+E_v(t)(t-a))-u(a)v(a)}{t-a}$$

$$=\lim_{t\to a}\frac{u'(a)(t-a)v(a)+u(a)v'(a)(t-a)+(t-a)(u(a)E_v(t)+v(a)E_u(t))+(t-a)^2(u'(a)+E_u(t))(v'(a)+E_v(t))}{t-a}$$

$$=\lim_{t\to a}u'(a)v(a)+u(a)v'(a)+u(a)E_v(t)+v(a)E_u(t)+(t-a)(u'(a)+E_u(t))(v'(a)+E_v(t))$$

$$=\lim_{t\to a}u'(a)v(a)+u(a)v'(a).$$

39. Sample answer (other answers are possible):

If $F(t)=\sqrt[3]{t}$, $G(t)=1+t$, $H(t)=\sqrt[4]{t}$, $J(t)=t^2-1$, then $h(t)=F(G(H(J(x))))=\sqrt[3]{1+\sqrt[4]{t^2-1}}$.

41. The calculator form of $2^{2^{t^2+1}}$ is given by $2\text{^}(2\text{^}((t\text{^}2)+1))$.

Section 2.3 Differentiating Compositions

1. If $g(x)=2x+2$ and $h(x)=x^{13/3}$, then $f(x)=g(h(x))=2x^{13/3}+2$. So, by the chain rule, we have $f'(x)=\dfrac{d}{dx}g(h(x))=g'(h(x))h'(x)$.

Since $g'(x)=2$ and $h'(x)=\dfrac{13}{3}x^{(13/3-1)}=\dfrac{13}{3}x^{10/3}$ it follows that $f'(x)=2\left(\dfrac{13}{3}x^{10/3}\right)=\dfrac{26}{3}x^{10/3}$.

3. Let $y=u^{23}$ and $u=2x^2-3x+1$. By the chain rule, we have $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$.

Since $\dfrac{dy}{du}=\dfrac{d}{du}u^{23}=23u^{22}$ and $\dfrac{du}{dx}=\dfrac{d}{dx}(2x^2-3x+1)=4x-3$, it follows that

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=(23u^{22})(4x-3).$$

Substitution produces

$$\frac{dy}{dx}=23(2x^2-3x+1)^{22}(4x-3).$$

5. If $f(x)=x^{5/4}$ and $h(x)=x^3-5x+\dfrac{1}{2}$, then $g(x)=f(h(x))=\left(x^3-5x+\dfrac{1}{2}\right)^{5/4}$. So, by the chain rule, we have $g'(x)=\dfrac{d}{dx}f(h(x))=f'(h(x))h'(x)$.

Since $f'(x)=\dfrac{5}{4}x^{1/4}$ and $h'(x)=3x^2-5$, it follows that $g'(x)=\dfrac{5}{4}\left(x^3-5x+\dfrac{1}{2}\right)^{1/4}(3x^2-5)$.

7. Let $y = \sqrt{u} = u^{1/2}$ and $u = u(t) = 2t - 7$. Then $f(t) = y(u(t)) = \sqrt{2t-7}$. By the chain rule, we have $f'(t) = \dfrac{dy}{du}\dfrac{du}{dt}$.

Since $\dfrac{dy}{du} = \dfrac{d}{du}u^{1/2} = \dfrac{1}{2}u^{-1/2}$ and $\dfrac{du}{dt} = \dfrac{d}{dt}(2t-7) = 2$, it follows that

$$\begin{aligned} f'(t) &= \frac{dy}{dt} \\ &= \frac{dy}{du}\frac{du}{dt} \\ &= \left(\frac{1}{2}u^{-1/2}\right)(2) \\ &= \frac{1}{\sqrt{u}}. \end{aligned}$$

Substitution produces

$$f'(t) = \frac{1}{\sqrt{2t-7}}.$$

9. By the product rule,

$$\begin{aligned} &\frac{dr}{d\theta} \\ &= [(2\theta-1)^6]'(4\theta+\pi)^8 + (2\theta-1)^6[(4\theta+\pi)^8]'. \end{aligned}$$

To complete the problem we apply the chain rule to $[(2\theta-1)^6]'$ and $[(4\theta+\pi)^8]'$.

First, regard $(2\theta-1)^6$ as u^6 where $u = 2\theta - 1$. Then $[(2\theta-1)^6]' = 6u^5\dfrac{du}{d\theta} = 6(2\theta-1)^5(2)$.

Treating $(4\theta+\pi)^8$ as v^8 where $v = 4\theta + \pi$ we find that $[(4\theta+\pi)^8]' = 8v^7\dfrac{dv}{d\theta} = 8(4\theta+\pi)^7(4)$.

Finally,

$$\begin{aligned} \frac{dr}{d\theta} &= 12(2\theta-1)^5(4\theta+\pi)^8 \\ &= (2\theta-1)^6(32)(4\theta+\pi)^7. \end{aligned}$$

11. Let $g(x) = \sqrt{1+x}$. Then

$$y = g(g(g(x))) = \sqrt{1+\sqrt{1+\sqrt{1+x}}}.$$

By the chain rule, it follows that

$$\frac{dy}{dx} = g'(g(g(x)))g'(g(x))g'(x).$$

Since

$$\begin{aligned} g'(x) &= \frac{d}{dx}(1+x)^{1/2} \\ &= \frac{1}{2}(1+x)^{-1/2} \\ &= \frac{1}{2}\frac{1}{\sqrt{1+x}} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= g'(g(g(x)))g'(g(x))g'(x) \\ &= \frac{1}{2\sqrt{1+g(g(x))}}\frac{1}{2\sqrt{1+g(x)}}\frac{1}{2\sqrt{1+x}} \\ &= \frac{1}{8}\frac{1}{\sqrt{1+\sqrt{1+\sqrt{1+x}}}}\frac{1}{\sqrt{1+\sqrt{1+x}}}\frac{1}{\sqrt{1+x}}. \end{aligned}$$

13. If $g(u) = \sqrt[3]{u} = u^{1/3}$ and $f(u) = \dfrac{4u-1}{3u+7}$, then $h(u) = g(f(u)) = \sqrt[3]{\dfrac{4u-1}{3u+7}}$. So, by the chain rule, we have

$$h'(u) = \frac{d}{du}g(f(u)) = g'(f(u))f'(u).$$

Since $g'(u) = \dfrac{1}{3}u^{-2/3}$ and

$$\begin{aligned} f'(u) &= \frac{(4u-1)'(3u+7) - (4u-1)(3u+7)'}{(3u+7)^2} \\ &= \frac{(4)(3u+7) - (4u-1)(3)}{(3u+7)^2} \\ &= \frac{31}{(3u+7)^2}, \end{aligned}$$

it follows that

$$\begin{aligned} h'(u) &= g'(f(u))f'(u) \\ &= \frac{1}{3}\left(\frac{4u-1}{3u+7}\right)^{-2/3}\frac{31}{(3u+7)^2}. \end{aligned}$$

15. Here we have

$$H(z) = (3z^2 - 1)^4 (4z^{-3} + 2z^{-2} + 4)\sqrt{1 - \frac{1}{z}}.$$

So,

$$H'(z) = [(3z^2 - 1)^4]'(4z^{-3} + 2z^{-2} + 4)\sqrt{1 - \frac{1}{z}} + (3z^2 - 1)^4 (4z^{-3} + 2z^{-2} + 4)'\sqrt{1 - \frac{1}{z}}$$
$$+ (3z^2 - 1)^4 (4z^{-3} + 2z^{-2} + 4)\left(\sqrt{1 - \frac{1}{z}}\right)'.$$

We now compute the three derivatives in the above expression.

(i) Let $y = u^4$ and $u = 3z^2 - 1$. Then $\frac{dy}{dx} = 4u^3$, $\frac{du}{dz} = 6z$ and so

$$\begin{aligned}[(3z^2 - 1)^4]' &= \frac{dy}{du}\frac{du}{dz} \\ &= (4u^3)(6z) \\ &= 24z(3z^2 - 1)^3.\end{aligned}$$

(ii) $$\begin{aligned}(4z^{-3} + 2z^{-2} + 4)' &= (4z^{-3})' + (2z^{-2})' + (4)' \\ &= (-12)z^{-4} + (-4)z^{-3}\end{aligned}$$

(iii) Let $f(z) = \sqrt{z}$ and $g(z) = 1 - \frac{1}{z}$. Then $f'(z) = \frac{1}{2\sqrt{z}}$, $g'(z) = 0 - (-1)z^{-2} = \frac{1}{z^2}$ and so

$$\begin{aligned}\left(\sqrt{1 - \frac{1}{z}}\right)' &= f'(g(z))g'(z) \\ &= \frac{1}{2\sqrt{1 - \frac{1}{z}}}\left(\frac{1}{z^2}\right)\end{aligned}$$

Substitution yields

$$\begin{aligned}H'(z) &= [24z(3z^2 - 1)^3](4z^{-3} + 2z^{-2} + 4)\sqrt{1 - \frac{1}{z}} \\ &+ (3z^2 - 1)^4(-12z^{-4} - 4z^{-3})\sqrt{1 - \frac{1}{z}} \\ &+ (3z^2 - 1)^4(4z^{-3} + 2z^{-2} + 4)\frac{1}{2\sqrt{1 - \frac{1}{z}}}\frac{1}{z^2}.\end{aligned}$$

17. Find $\frac{dy}{dx}$ if $y = f(f(x))$ where $f(x) = x^2 - 3x + 1$. So $f'(x) = 2x - 3$.

By the chain rule, $$\begin{aligned}\frac{dy}{dx} &= f'(f(x))\, f'(x) \\ &= (2f(x) - 3)(2x - 3) \\ &= (2(x^2 - 3x + 1) - 3)(2x - 3).\end{aligned}$$

19. Find $\frac{dy}{dx}$ if $y = u^2 + 1$ and $u = x^{-3/2} + 1$.
By the chain rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (u^2+1)'(x^{-3/2}+1)'\\ &= (2u)\left(\frac{-3}{2}x^{-5/2}\right)\\ &= -3ux^{-5/2}\\ &= -3(x^{-3/2}+1)x^{-5/2}.\end{aligned}$$

21. Find $\frac{dy}{dx}$ if $y = \sqrt{w} = w^{1/2}$, $w = \frac{1}{v} = v^{-1}$, $v = 2x - 1$.

By the chain rule, $\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dv}\frac{dv}{dx}$.

Since $\frac{dy}{dw} = \frac{1}{2}w^{-1/2}$, $\frac{dw}{dv} = -1v^{-2}$, $\frac{dv}{dx} = 2$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dw}\frac{dw}{dv}\frac{dv}{dx}\\ &= \left(\frac{1}{2}w^{-1/2}\right)(-v^{-2})(2).\end{aligned}$$

Substituting the given information we obtain the derivative

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{w^{1/2}v^2}\\ &= \frac{-1}{(v^{-1})^{1/2}v^2}\\ &= \frac{-1}{v^{3/2}}\\ &= \frac{-1}{(2x-1)^{3/2}}.\end{aligned}$$

23. Determine $\frac{dy}{dx}$ provided that $y = H(t^2+1)$, $t = 3x - 1$, and $H'(v) = \frac{1}{v}$. By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= H'(t^2+1)\ \frac{d}{x}(t^2+1)\\ &= H'(t^2+1)\left(\frac{d}{dt}(t^2+1)\right)\frac{dt}{dx}\\ &= \frac{1}{t^2+1}(2t)\ \frac{dt}{dx}\\ &= \frac{1}{(3x-1)^2+1}[2(3x-1)](3)\\ &= \frac{6(3x-1)}{(3x-1)^2+1}.\end{aligned}$$

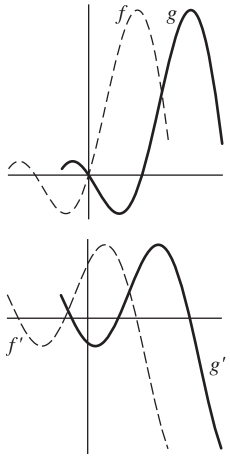
25. We find $f'(w) = \frac{d}{dx}|(x-2)(x+1)|$ using the chain rule and the fact that $\frac{d}{dx}|x| = \frac{x}{|x|}\ (x \neq 0)$.
Let $g(x) = |x|$ and
$h(x) = (x-2)(x+1) = x^2 - x - 2$. Then
$f(x) = g(h(x))$. Since $g'(x) = \frac{x}{|x|}\ (x \neq 0)$ and
$h'(x) = 2x - 1$, we have

$$\begin{aligned}f'(x) &= \frac{d}{dx}\ g(h(x))\\ &= g'(h(x))\ h'(x)\\ &= \frac{h(x)}{|h(x)|}\ h'(x)\\ &= \frac{x^2-x-2}{|x^2-x-2|}(2x-1).\end{aligned}$$

27. We find $\frac{dy}{dx} = \frac{d}{dx}\left(|x^3-x|+2\right)^3$ by using the chain rule and the fact that $\frac{d}{dx}|x| = \frac{x}{|x|}\ (x \neq 0)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(|x^3-x|+2\right)^3\\ &= 3\left(|x^3-x|+2\right)^2\ \frac{d}{dx}\left(|x^3-x|+2\right)\\ &= 3\left(|x^3-x|+2\right)^2\ \frac{x^3-x}{|x^3-x|}\ \frac{d}{dx}(x^3-x)\\ &= 3\left(|x^3-x|+2\right)^2\left(\frac{(x^3-x)(3x^2-1)}{|x^3-x|}\right)\end{aligned}$$

29.



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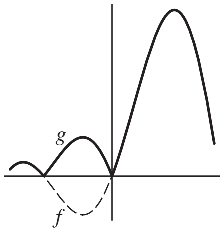
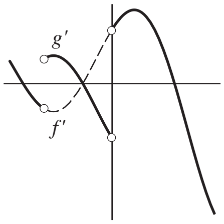




33. First we find $\left.\dfrac{dy}{dx}\right|_{x=0}$ of $y=\dfrac{1}{\sqrt{x^2+1}}=(x^2+1)^{-1/2}$. By the chain rule we have

$$\begin{aligned}\frac{dy}{dx}&=-\frac{1}{2}(x^2+1)^{-3/2}(2x)\\&=\frac{-x}{(x^2+1)^{3/2}}\end{aligned}$$

so $\left.\dfrac{dy}{dx}\right|_{x=1}=\dfrac{-1}{(1+1)^{3/2}}=-\dfrac{1}{2\sqrt{2}}.$

This is the slope of the tangent line. For the equation of the tangent line at $\left(1,\dfrac{1}{\sqrt{2}}\right)$ we get

$$y-\frac{1}{\sqrt{2}}=\frac{-1}{2\sqrt{2}}(x-1).$$

35. First we find $\left.\dfrac{dy}{dx}\right|_{x=2}$ of $y=\dfrac{(2x-3)^3}{(-x^2+x)^5}$ using the quotient rule and the chain rule.

$$\begin{aligned}\frac{dy}{dx}&=\frac{((2x-3)^3)'(-x^2+x)^5-(2x-3)^3((-x^2+x)^5)'}{((-x^2+x)^5)^2}\\&=\frac{3(2x-3)^2(2x-3)'(-x^2+x)^5-(2x-3)^3(5(-x^2+x)^4(-x^2+x)')}{(-x^2+x)^{10}}\\&=\frac{3(2x-3)^2(2)(-x^2+x)^5-(2x-3)^3(5(-x^2+x)^4(-2x+1))}{(-x^2+x)^{10}}\end{aligned}$$

so $\left.\dfrac{dy}{dx}\right|_{x=2}=\dfrac{3}{64}$ is the slope of the tangent line at $\left(2,\dfrac{-1}{32}\right)$. For the equation of the tangent line at $\left(2,\dfrac{-1}{32}\right)$ we get $y+\dfrac{1}{32}=\dfrac{3}{64}(x-2).$

37. We want to find $F'(t)\big|_{t=15}$ if $F = F(y)$ and $y(15) = 25.5$, $F(25.5) = -20$, $y'(15) = 4.25$, and $F'(25.5) = -1.9$.

Using the chain rule we get $F'(t) = (F(y))' = (F'(y))\ y'$ simplifying when $t = 15$ yields

$$F'(t)' = (F'(25.5))(4.25) = (-1.9)(4.25)$$
$$= -8.075°\text{F / sec.}$$

$y(15) = 25.5$ and $F(25.5) = -20$ tell us that when $t = 15$ the object is at 25.5 units on the y-axis and when the object is at 25.5 units its temperature is –20°F.

$y'(15) = 4.25$ and $F'(25.5) = -1.9$ tell us that when $t = 15$ the object is moving at a rate of 4.25 units/sec on the y-axis and when an object is at a height of 25.5 units its temperature is changing at a rate of –1.9°F/sec.

Section 2.4 Implicit Differentiation

1. To find $\dfrac{dy}{dx}$ from the equation $3x^2 - 4y^2 = 3$ we assume that y is a function of x and differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(3x^2 - 4y^2) = \frac{d}{dx}\ 3$$
$$\frac{d}{dx}\ 3x^2 - \frac{d}{dx}\ 4y^2 = \frac{d}{dx}\ 3$$
$$6x - 8y\ \frac{dy}{dx} = 0.$$

Solving this for $\dfrac{dy}{dx}$ yields $\dfrac{dy}{dx} = \dfrac{3x}{4y}$.

3. To find $\dfrac{dy}{dx}$ from the equation $3x - 2y + \dfrac{y}{x} - 7 = 0$ we assume that y is a function of x and differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}\left(3x - 2y + \frac{y}{x} - 7\right) = \frac{d}{dx}\ 0$$
$$\frac{d}{dx}\ 3x - \frac{d}{dx}\ 2y + \frac{d}{dx}\frac{y}{x} - \frac{d}{dx}\ 7 = \frac{d}{dx}\ 0$$
$$\frac{d}{dx}\ 3x - \frac{d}{dx}\ 2y + \frac{x\frac{d}{dx}\ y - y\dfrac{d}{dx}\ x}{x^2} - \frac{d}{dx}\ 7 = \frac{d}{dx}\ 0$$
$$3 - 2\ \frac{dy}{dx} + \frac{x\ \frac{dy}{dx} - y}{x^2} - 0 = 0.$$

Solving for $\dfrac{dy}{dx}$ yields $\dfrac{dy}{dx} = \dfrac{3x^2 - y}{2x^2 - x}$.

5. Assuming that y is a function of x, we seek $\dfrac{dy}{dx}$ given the relation $2x^2 - 3xy + y^2 = 2$.

We begin by differentiating both sides of the relation with respect to the independent variable x keeping in mind that $y = y(x)$.

$$\frac{d}{dx}(2x^2 - 3xy + y^2) = \frac{d}{dx}\ 2$$
$$\frac{d}{dx}(2x^2) - \frac{d}{dx}(3xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}\ 2$$
$$4x - 3\left(\frac{dx}{dx}\ y + x\frac{dy}{dx}\right) + 2y\ \frac{dy}{dx} = 0$$

We solve the last equation for $\dfrac{dy}{dx}$ obtaining

$$\frac{dy}{dx} = \frac{3y - 4x}{2y - 3x}.$$

7. Regarding y as a function of x, we seek $\dfrac{dy}{dx}$ given the relation $(x - 2y)^2 - xy^2 = -2xy + 4$. We first simplify the relation by expanding $(x - 2y)^2$ and then writing the terms involving y on one side of the equal sign obtaining $4y^2 - xy^2 - 2xy = 4 - x^2$. We now differentiate both sides of this last equation with respect to x and simplify:

$$\frac{d}{dx}(4y^2 - xy^2 - 2xy) = \frac{d}{dx}(4 - x^2)$$
$$4\frac{d}{dx}(y^2) - \frac{d}{dx}(xy^2) - 2\frac{d}{dx}(xy) = \frac{d}{dx}(4) - \frac{d}{dx}(x^2)$$
$$4\left(2y\frac{dy}{dx}\right) - \left(y^2 + x\left(2y\frac{dy}{dx}\right)\right) - 2\left(y + x\frac{dy}{dx}\right) = -2x$$
$$8y\frac{dy}{dx} - 2xy\frac{dy}{dx} - 2x\frac{dy}{dx} = y^2 + 2y - 2x$$

Solving the last equation for $\dfrac{dy}{dx}$ produces:

$$\frac{dy}{dx} = \frac{y^2 + 2y - 2x}{2(4y - xy - x)}.$$

9. Determine $\frac{dy}{dx}$ if $y = y(x)$ and we have the relation $\frac{3x^2+4y}{2x-6y^2} = x + y$.

Before differentiating we rewrite the relation as $3x^2+4y=(2x-6y^2)(x+y)$ or $3x^2+4y=2x^2-6xy^2+2xy-6y^3$. Then we differentiate the left side and right side of the above relation.

$$\frac{d}{dx}(3x^2+4y)=\frac{d}{dx}(2x^2-6xy^2+2xy-6y^3)$$

$$\frac{d}{dx}(3x^2)+\frac{d}{dx}(4y)=\frac{d}{dx}(2x^2)-\frac{d}{dx}(6xy^2)+\frac{d}{dx}(2xy)-\frac{d}{dx}6y^3$$

$$6x+4\frac{dy}{dx}=4x-6y^2-12xy\frac{dy}{dx}+2y+2x\frac{dy}{dx}-18y^2\frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx}=\frac{6y^2+2x-2y}{2x-18y^2-12xy-4}=\frac{3y^2+x-y}{x-9y^2-6xy-2}$.

It is also possible to differentiate the original relation (without clearing fractions). In that case we get the equally correct $\frac{dy}{dx}=\frac{-x^2+6xy^2+18y^4+4y}{4x+12y^2+18x^2y-2x^2+12xy^2-18y^4}$.

11. Since $4(1)^2-(2)(1)+2(2)-3(1)=3$ the point $P = (2, 1)$ is on the graph of $4y^2-xy+2x-3y=3$. We wish to determine the value of $\left.\frac{dy}{dx}\right|_{x=2,\ y=1}$ given that $4y^2-xy+2x-3y=3$. We differentiate both sides of the relation with respect to x, regarding y as a function of x:

$$\frac{d}{dx}(4y^2-xy+2x-3y)=\frac{d}{dx}3$$

$$4\frac{d}{dx}(y^2)-\frac{d}{dx}(xy)+2\frac{d}{dx}x-3\frac{d}{dx}y=\frac{d}{dx}3$$

$$4\frac{d}{dx}(y^2)-\left(\left(\frac{d}{dx}x\right)y+x\left(\frac{d}{dx}y\right)\right)+2\frac{d}{dx}x-3\frac{d}{dx}y=\frac{d}{dx}3$$

$$4\left(2y\frac{dy}{dx}\right)-\left(y+x\frac{dy}{dx}\right)+2-3\frac{dy}{dx}=0.$$

Solving the last equation for $\frac{dy}{dx}$ yields $\frac{dy}{dx}=\frac{y-2}{8y-x-3}$.

Substituting 2 for x and 1 for y in $\frac{dy}{dx}$ produces $\left.\frac{dy}{dx}\right|_{x=2,\ y=1}=\frac{-1}{3}$.

13. Since $\frac{1}{(1)+(-2)}+3-4((1)+(-2))^2=-2$ the point P = (1, –2) is on the graph of $\frac{1}{x+y}+3-4(x+y)^2=-2$. We wish to determine the value of $\left.\frac{dy}{dx}\right|_{x=1,\ y=-2}$ given that $\frac{1}{x+y}+3-4(x+y)^2=-2$. We differentiate both sides of the given relation with respect to x regarding y as a function of x:

$$\frac{d}{dx}\left(\frac{1}{x+y}+3-4(x+y)^2\right)=\frac{d}{dx}(-2)$$

$$\frac{d}{dx}[(x+y)^{-1}]+\frac{d}{dx}3-4\frac{d}{dx}[(x+y)^2]=\frac{d}{dx}(-2)$$

$$(-1)(x+y)^{-2}\frac{d}{dx}(x+y)-4\left[2(x+y)\frac{d}{dx}(x+y)\right]=0$$

$$\frac{-1}{(x+y)^2}\left(\frac{d}{dx}x+\frac{d}{dx}y\right)-8(x+y)\left(\frac{d}{dx}x+\frac{d}{dx}y\right)=0$$

$$\frac{-1}{(x+y)^2}\left(1+\frac{dy}{dx}\right)-8(x+y)\left(1+\frac{dy}{dx}\right)=0.$$

Solving the last equation for $\frac{dy}{dx}$ yields $\frac{dy}{dx}=-1$ for all x and y in the domain of the derivative. So, in particular, $\left.\frac{dy}{dx}\right|_{x=1,\ y=-2}=-1.$

15. We first find the value of $\left.\frac{dy}{dx}\right|_{x=1,\ y=1}$ for the equation $xy-x+3y=3$. We differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(xy-x+3y)=\frac{d}{dx}(3)$$

$$\frac{d}{dx}(xy)-\frac{d}{dx}(x)+\frac{d}{dx}(3y)=\frac{d}{dx}(3)$$

$$y+x\frac{dy}{dx}-1+3\frac{dy}{dx}=0$$

Solving for $\frac{dy}{dx}$ yields $\frac{dy}{dx}=\frac{1-y}{x+3}$.

So $\left.\frac{dy}{dx}\right|_{x=1,\ y=1}=0$. Since at (1, 1) the slope of the tangent line is 0 we know that the equation for the tangent line is y = 1.

17. We first find the value of $\left.\frac{dy}{dx}\right|_{x=-1,\ y=1}$ for the equation $\frac{x^2+y^2}{2x+3y}=-2x$.

We differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}\left(\frac{x^2+y^2}{2x+3y}\right)=\frac{d}{dx}(-2x)$$

$$\frac{(2x+3y)\frac{d}{dx}(x^2+y^2)-(x^2+y^2)\frac{d}{dx}(2x+3y)}{(2x+3y)^2}=-2$$

$$\frac{(2x+3y)\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)\right]-(x^2+y^2)\left[\frac{d}{dx}(2x)+\frac{d}{dx}(3y)\right]}{(2x+3y)^2}=-2$$

$$\frac{(2x+3y)\left(2x+2y\frac{dy}{dx}\right)-(x^2+y^2)\left(2+3\frac{dy}{dx}\right)}{(2x+3y)^2}=-2$$

solving for $\frac{dy}{dx}$ yields $\frac{dy}{dx}=\frac{10x^2+30xy+16y^2}{3x^2-4xy-3y^2}$. So $\left.\frac{dy}{dx}\right|_{x=-1,\ y=1}=-1$.

We can now find the equation for the tangent line at (–1, 1).

$y-1=-1(x+1)$ or $y=-x$

19. First we solve $x^2+y^2-2x+4y=4$ for y.

$$y^2+4y=4-x^2+2x$$

$$(y+2)^2=8-x^2+2x$$

$$y=-2\pm\sqrt{8-x^2+2x}$$

Because the darkened portion is the bottom half of the circle, we get $y=-2-\sqrt{8-x^2+2x}$.

21. First we solve $2x^2-10xy+4y^2=3$ for y.

Using the quadratic formula for $4y^2-10xy+2x^2-3$ yields

$$y=\frac{5x\pm\sqrt{17x^2+12}}{4}.$$

Because the darkened portion is the upper portion of the hyperbola, we get

$$y=\frac{5x+\sqrt{17x^2+12}}{4}.$$

23. We assume that $x^2+y^2\neq 0$ and then simplify

$$\frac{x}{x^2+y^2}=4.$$

$$\frac{x}{x^2+y^2}=4$$

$$x^2+y^2=\frac{x}{4}$$

$$\left(x-\frac{1}{8}\right)^2+y^2=\frac{1}{64}$$

This is the equation for the circle with center $\left(\frac{1}{8},\ 0\right)$ and radius $\frac{1}{8}$ (except for the point (0, 0)).

25. Using implicit differentiation with respect to x on $4x^2+4y^2-x=0$ yields

$$(4x^2+4y^2-x)'=(0)'$$

$$8x+8yy'-1=0.$$

Solving for y' we get $y'=\frac{1-8x}{8y}(y\neq 0)$.

To show $\frac{1-8x}{8y}$ is equivalent to $\frac{y^2-x^2}{2xy}$ we set them equal to each other and simplify:

$$\frac{1-8x}{8y}=\frac{y^2-x^2}{2xy}$$

$$\frac{1-8x}{4}=\frac{y^2-x^2}{x}$$

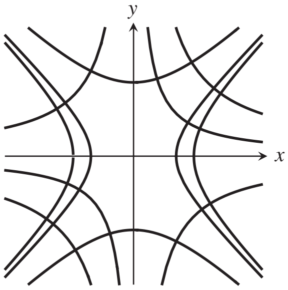
$$x-8x^2=4y^2-4x^2$$

$$x=4y^2+4x^2$$

$$\frac{x}{x^2+y^2}=4$$

and this equation holds for all (x, y) on the curve.

27.



We differentiate with respect to x to find the slope of the tangent lines of $x^2 - y^2 = c$, obtaining $\frac{dy}{dx} = \frac{x}{y}$.

Similarly, we get the slope of the tangent lines of $xy = d$, obtaining $\frac{dy}{dx} = \frac{-y}{x}$. These two slopes are orthogonal.

29. a. Assuming that $y = y(x)$, find $\dfrac{dy}{dx}$ given the relation $2xy^3 + x^2y^2 - 3y + x = 1$. We differentiate both sides of the above equation with respect to x:

$$(2xy^3 + x^2y^2 - 3y + x)' = (1)'$$
$$(2xy^3)' + (x^2y^2)' - (3y)' + (x)' = (1)'$$
$$2(x'y^3 + x(y^3)') + (x^2)'y^2 + x^2(y^2)' - 3(y') + x' = (1)'$$
$$2(y^3 + 3xy^2y') + 2xy^2 + 2x^2yy' - 3y' + 1 = 0.$$

Solving the last equation for $y' = \dfrac{dy}{dx}$ we obtain $y' = \dfrac{dy}{dx} = \dfrac{1 + 2xy^2 + 2y^3}{3 - 6xy^2 - 2x^2y}$.

b. Assuming that $x = x(y)$, find $\dfrac{dx}{dy}$ given the relation $2xy^3 + x^2y^2 - 3y + x = 1$. We differentiate both sides of the above equation with respect to y:

$$(2xy^3 + x^2y^2 - 3y + x)' = (1)'$$
$$(2xy^3)' + (x^2y^2)' - (3y)' + (x)' = (1)'$$
$$2(x'y^3 + x(y^3)') + (x^2)'y^2 + x^2(y^2)' - 3(y') + x' = (1)'$$
$$2(x'y^3 + 3xy^2) + 2xx'y^2 + 2x^2y - 3 + x' = 0$$

Solving the last equation for $x' = \dfrac{dy}{dx}$ we obtain $x' = \dfrac{dx}{dy} = \dfrac{3 - 6xy^2 - 2x^2y}{1 + 2xy^2 + 2y^3}$.

c. From parts a and b, it is clear that $\left(\dfrac{dy}{dx}\right)^{-1} = \dfrac{dx}{dy}$ and so $\dfrac{dy}{dx}\dfrac{dx}{dy} = 1$. To explain this relationship between $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ graphically we consider two near-by points, say (x_1, y_1) and (x_2, y_2), on the curve described by $2xy^3 + x^2y^2 - 3y + x = 1$. Then we have

$\dfrac{dy}{dx} \approx \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $\dfrac{dx}{dy} \approx \dfrac{\Delta x}{\Delta y} = \dfrac{x_2 - x_1}{y_2 - y_1}$.

Again, we see that $\dfrac{dy}{dx}\dfrac{dx}{dy} = 1$.

31. a. The ellipse $x^2 - xy + \frac{3}{4}y^2 = 7$ crosses the x-axis when $y = 0$ and the y-axis when $x = 0$. If $y = 0$, then the equation becomes $x^2 = 7$ and so $x = \pm\sqrt{7}$. If $x = 0$, then the equation for the ellipse becomes $\frac{3}{4}y^2 = 7$ and so $y = \pm\sqrt{\frac{28}{3}} = \pm 2\sqrt{\frac{7}{3}}$. Hence, the ellipse crosses the coordinate axes at $(\pm\sqrt{7}, 0)$ and $\left(0, \pm 2\sqrt{\frac{7}{3}}\right)$.

b. We seek points on the graph of $x^2 - xy + \frac{3}{4}y^2 = 7$ where the tangent line is horizontal. That is, we seek (x, y) so that $\dfrac{dy}{dx} = 0$. To compute $\dfrac{dy}{dx}$ we first differentiate both sides of $x^2 - xy + \frac{3}{4}y^2 = 7$ with respect to x:

$$\frac{d}{dx}\left(x^2 - xy + \frac{3}{4}y^2\right) = \frac{d}{dx}(7)$$
$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}\left(\frac{3}{4}y^2\right) = \frac{d}{dx}(7)$$
$$\frac{d}{dx}(x^2) - \left(\left(\frac{d}{dx}x\right)y + x\left(\frac{d}{dx}y\right)\right) + \frac{3}{4}\left(\frac{d}{dx}y^2\right) = \frac{d}{dx}(7)$$
$$(2x) - \left((1)y + x\frac{dy}{dx}\right) + \frac{3}{4}\left(2y\frac{dy}{dx}\right) = 0.$$

Solving the last equation for $\frac{dy}{dx}$ we have $\frac{dy}{dx} = \frac{2(y-2x)}{3y-2x}$. We see that $\frac{dy}{dx} = 0$ at points on the ellipse where $y = 2x$. If $y = 2x$, then substitution into the equation for the ellipse yields the equation $2x^2 = 7$. Hence, the ellipse has horizontal tangent lines at $\left(\sqrt{\frac{7}{2}}, 2\sqrt{\frac{7}{2}}\right)$ and $\left(-\sqrt{\frac{7}{2}}, -2\sqrt{\frac{7}{2}}\right)$.

c. The derivative $\frac{dy}{dx} = \frac{2(y-2x)}{3y-2x}$ fails to exist at points on the ellipse where $y = \frac{2}{3}x$. If $y = \frac{2}{3}x$, then the equation for the ellipse becomes $2x^2 = 21$. Hence, the ellipse has vertical tangent lines at the points $\left(\sqrt{\frac{21}{2}}, \frac{2}{3}\sqrt{\frac{21}{2}}\right)$ and $\left(-\sqrt{\frac{21}{2}}, -\frac{2}{3}\sqrt{\frac{21}{2}}\right)$.

Section 2.5 Trigonometric Functions

1. Find $\frac{dy}{dx}$ if $y = \sin(2x+3)$. We begin by decomposing $y = \sin(2x+3)$ into composition chain:

$y = \sin u$

$u = 2x + 3.$

We then have

$$\frac{dy}{du} = (\sin u)' = \cos u$$
$$\frac{du}{dx} = (2x+3)' = 2.$$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(2).$$

Substitute $u = 2x + 3$ into the last expression to get $\frac{dy}{dx} = 2\cos(2x+3)$.

3. Find $f'(t)$ if $f(t) = \tan(3t^2 - 4t + 1)$. We use the chain rule.

$$\begin{aligned} f'(t) &= (\tan(3t^2 - 4t + 1))' \\ &= \sec^2(3t^2 - 4t + 1)(3t^2 - 4t + 1)' \\ &= (6t - 4)\sec^2(3t^2 - 4t + 1) \end{aligned}$$

5. Find $f'(x)$ if $f(x)=-\dfrac{\cos x}{x}$. By the quotient rule,

$$\begin{aligned} f'(x) &= \left(-\frac{\cos x}{x}\right)' \\ &= -\frac{(\cos x)'x-(\cos x)(x)'}{x^2} \\ &= -\frac{(-\sin x)x-(\cos x)(1)}{x^2} \\ &= \frac{x\sin x+\cos x}{x^2}. \end{aligned}$$

7. Compute $T'(\theta)$ if $T(\theta)=\tan\left(\dfrac{1+\theta}{1-\theta}\right)$. If $f(\theta)=\tan\theta$ and $g(\theta)=\dfrac{1+\theta}{1-\theta}$, then $T(\theta)=(f\circ g)(\theta)=f(g(\theta))$.

We next find

$$\begin{aligned} f'(\theta) &= (\tan\theta)' \\ &= \sec^2\theta \end{aligned}$$

and

$$\begin{aligned} g'(\theta) &= \left(\frac{1+\theta}{1-\theta}\right)' \\ &= \frac{(1+\theta)'(1-\theta)-(1+\theta)(1-\theta)'}{(1-\theta)^2} \\ &= \frac{(1)(1-\theta)-(1+\theta)(-1)}{(1-\theta)^2} \\ &= \frac{2}{(1-\theta)^2}. \end{aligned}$$

Hence, by the chain rule,

$$\begin{aligned} T'(\theta) &= (f\circ g)'(\theta)=f'(g(\theta))g'(\theta) \\ &= (\sec^2 g(\theta))g'(\theta) \\ &= \left(\sec^2\left(\frac{1+\theta}{1-\theta}\right)\right)\frac{2}{(1-\theta)^2}. \end{aligned}$$

9. Find $f'(x)$ if $f(x)=2\sin(x^2+1)$. If $g(x)=2\sin x$, and $h(x)=x^2+1$ then $f(x)=g(h(x))$.

We next find $g'(x)=2\cos x$ and $h'(x)=2x$

Hence, by the chain rule,

$$\begin{aligned} f'(x) &= g'(h(x))(h'(x)) \\ &= 2\cos(x^2+1)(2x) \\ &= 4x\cos(x^2+1). \end{aligned}$$