T **27.** Let L be the value of the limit in Exercise 22. Find a $\delta > 0$ so that when

$$0 < |\theta - \pi/3| < \delta$$

$$\text{we have} \quad \left| \frac{\cos\ \theta - 1/2}{\theta - \pi/3} - L \right| < 0.01.$$

T **28.** Let L be the value of the limit in Exercise 24. Find a $\delta > 0$ so that when

$$0 < |x - 10| < \delta$$

$$\text{we have} \quad \left| \frac{\log_{10} x - 1}{x - 10} - L \right| < 0.01.$$

Exercises 29–32: Find $f'(x)$ for the given f.

29. $f(x) = 2x - 5$

30. $f(x) = -3x^3 + 4x$

31. $f(x) = x + \dfrac{1}{x}$

32. $f(x) = (x + a)^2$, a constant.

T **33.** Find the equation for the line tangent to $y = \sqrt{x + 2}$ at the point $(7, 3)$. Find $r > 0$ so that for $7 - r < x < 7 + r$ the error in the tangent line approximation to $\sqrt{x + 2}$ is no more than 0.001 in absolute value.

T **34.** Use graphical or numerical methods to find the derivative of $f(x) = \cos x$ at $x = \pi/2$. Find an equation for the line tangent to $y = \cos x$ at the point $(\pi/2, 0)$. Find $r > 0$ so that for $\pi/2 - r < x < \pi/2 + r$ the error in the tangent line approximation to $\cos x$ is no more than 0.005 in absolute value.

35. Write a short paragraph that clearly explains what is meant by the statement

$$\lim_{x \to a} f(x) \quad \text{does not exist.}$$

Illustrate your explanation with some graphs.

36. In Section 1.7 we showed that if a function f has a derivative at $x = a$, then it is continuous at $x = a$. Is the converse true? That is, if a function f is continuous at $x = a$, must it have a derivative at $x = a$? Justify your answer.

37. In Section 1.7 we showed that if a function f has a derivative at $x = a$, then it is continuous at $x = a$. Explain why if f is not continuous at $x = a$, then we will never see a "straight line" when we zoom in on the graph of $y = f(x)$ at the point $(a, f(a))$.

STUDENT PROJECT

EXPLORING CHAOS

What Is Chaos?

Meteorologists say that "a butterfly flapping its wings in China can cause a tornado in Kansas several weeks later." Although it is very unlikely that the truth of this statement will ever actually be demonstrated, the statement does illustrate a problem with long-range weather prediction. Very small disturbances in the atmosphere can, over time, lead to large, noticeable phenomena. To accurately predict the weather a month from today, meteorologists would need precise information (pressure, humidity, velocity, temperature, etc.) now, and they would need this information about every cubic foot of the atmosphere. And even if it were possible to get information on this scale, small errors in some of these measurements might be enough to make weather patterns differ drastically from predictions.

The atmosphere is an example of a *chaotic system.* A chaotic system is one in which very small changes in one part of the system can lead to very big changes in another part. If the butterfly in China does not flap its wings at 2:00 P.M. on April 21, 2002, then it will be a nice day in Kansas on May 30. But if the butterfly does flap its wings, this small change in the atmosphere could propagate and eventually lead to a tornado on May 30. In recent years, mathematicians, scientists, and engineers have come to realize that many natural phenomena once thought to be well understood are really chaotic systems.

Compositions and Chaos In this project we will look at a fascinating chaotic system that arises by repeatedly composing the polynomial

$$p(x) = cx(1 - x)$$

with itself. Pick a number c with $0 < c < 4$. The system we examine will be the list of numbers we get by setting $x = 0.3$ and computing

$$p(0.3),\ p(p(0.3)),\ p(p(p(0.3))),\ p(p(p(p(0.3)))),\dots. \tag{1}$$

(The choice $x = 0.3$ is arbitrary. Any x with $0 < x < 1$ would work as well.) In constructing (1) we use $x = 0.3$ as input for the polynomial p. We get output $p(0.3)$. We then use this output as input for p, and so get as a next output $p(p(0.3))$. Now we use this output as input for p, and so on. Repeat this process several times. What happens to the list (1)? We will see that the answer depends on the value of c, and that very small changes in c can cause substantial changes in the list (1).

PROBLEM 1 Let $c = 2.1$, so

$$p(x) = 2.1x(1 - x).$$

Use a calculator or CAS to compute the first 10 terms of the list (1). Describe what happens to the numbers in the list. Next let $c = 2.5$, so

$$p(x) = 2.5x(1 - x).$$

Compute the first 15 terms of the list (1). Describe what happens to the numbers in the new list. How is the second list similar to the first list? How is it different?

PROBLEM 2 The number 0.52381 should be familiar from Problem 1. Again setting $c = 2.1$, compute $p(0.52381)$. The number 0.52381 is called a *fixed point* for $p(x)$. Explain why this terminology is appropriate. The exact value of the fixed point can be found by solving the equation

$$p(x) = x.$$

Explain why a solution of this equation is a fixed point of p. Find the exact value of the fixed point that is approximated by 0.52381. Repeat for the number 0.6 with $c = 2.5$.

PROBLEM 3 Now let $c = 3.4$ so

$$p(x) = 3.4x(1 - x).$$

Use a calculator or CAS to compute the first 100 terms of the list (1). Describe what happens to the numbers in the list. Repeat with $c = 3.5$, $c = 3.83$, $c = 3.845$, and $c = 3.862$. How do the lists obtained differ?

PROBLEM 4 In the previous problem you found that the values in the list eventually start to repeat. For $c = 3.4$ the list (1) eventually settled down to alternate between the two values 0.451963 and 0.842154. Explain why these two numbers are fixed points of the polynomial $p(p(x))$. Explain why these two numbers must be (approximate) solutions to the equation

$$p(p(x)) = x.$$

For $c = 3.5$, $c = 3.83$, and $c = 3.845$, list (1) also settled down to cycle repeatedly through a few values. What are these values? For which polynomial are these values fixed points?

Where's the Chaos? At the time of this writing, there does not seem to be a universally accepted definition of chaos. However, most of the proposed definitions state that if there is chaos, then it must be true that small changes in the input to a system can result in large changes in the output. We saw hints of such behavior in the previous problems. In working these problems we found that the behavior of list (1) changes as c changes. Sometimes the list settles down to one number; sometimes it cycles repeatedly between two, three, four, or six numbers; and sometimes there appears to be no eventual pattern. In addition, we saw that in some cases a very small change in c substantially changed the behavior of the list. Thus the list (1) exhibits chaos. In particular, very small changes in the number c can change the list from one that cycles through a few numbers to one with no apparent pattern. Indeed, it can be shown that for any positive integer n, there is a value of c between 0 and 4 such that the list (1) eventually settles down to cycle through n different values. In addition, as c gets closer and closer to 4 (but remains less than 4), it takes smaller and smaller changes in c to effect substantial changes in (1).

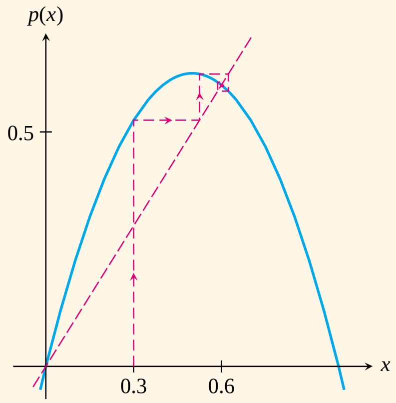

FIGURE 1.82 When $c = 2.5$, the list of compositions quickly closes in on 0.6.

Graphing the Composition List By using graphs, we can obtain a different view of how list (1) differs for different values of c. Since (1) consists of compositions, we can graph the compositions as we did in Section 1.1. For example, take $c = 2.5$, and then draw the graph of $y = p(x)$ and the line $y = x$. See Fig. 1.82. Starting from 0.3 on the x-axis, move up to the graph of p. We meet the graph of p at the point $(0.3, p(0.3))$. Change the output $p(0.3)$ to input by moving horizontally to the line $y = x$. We meet the line in the point $(p(0.3), p(0.3))$. Next move vertically to the graph of p again. We meet the graph in a point with y-coordinate $p(p(0.3))$. Continue the process just described, moving horizontally to the line $y = x$, and then vertically to the graph of p. Each time we meet p, the y-coordinate is the next element in the list (1). A programmable graphing calculator or CAS can be programmed to produce the resulting composition diagram very quickly. See Figs. 1.82 and 1.83.

For another way to investigate the behavior of (1), form the collection of ordered pairs

$$(0, 0.3), (1, p(0.3)), (2, p(p(0.3))), (3, p(p(p(0.3)))), \ldots,$$

where the first coordinate of an ordered pair tells how many times p was used in obtaining the second coordinate. When we plot several of these points (say, the first 200), we can get an idea of whether or not (1) settles down to cycle through some values. See Figs. 1.84 and 1.85.

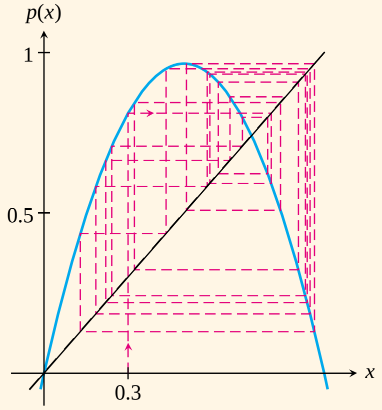

FIGURE 1.83 When $c = 3.862$, the list of compositions does not seem to enter a cycle.

PROBLEM 5 Generate graphs like those shown in Figs. 1.82, 1.83, 1.84, and 1.85 for three other values of c.

A Picture of Chaos To get a better picture of the chaotic nature of (1), we collect data on the behavior of the list for many values of c. For a given value of c, compute the list (1) until it seems to start cycling. For each number b in a cycle, construct the ordered pair (c, b). For example, when $c = 3.5$, list (1) eventually cycles through the four numbers 0.38282, 0.826941, 0.500884, and 0.874997. Thus we collect the four ordered pairs

$$(3.5, 0.38282), (3.5, 0.826941), (3.5, 0.500884), \text{ and } (3.5, 0.874997).$$

Do this for hundreds of values of c between 2 and 4 and then plot all of the points collected. Of course, there are many values of c for which (1) does not cycle, and others for which the cycle values become apparent only after many, many iterations. However, if we carry the list (1) to 1000 places for each of several hundred values of c, collect the last 100 entries in each list, and then use these 100 entries as the second coordinate in ordered pairs with first coordinate c, we obtain the graph shown in Fig. 1.86.

In this graph the c values run along the horizontal axis. For a given c value, the point or points above indicate entries 901 through 1000 in (1). In this picture we can see evidence of c values that result in cycles of length 2, 4, 8, and 3 as well as other c values that suggest more erratic behavior. This intriguing picture is also a fractal. Draw any small square in the graph with its right edge on the line $c = 4$. If we were to magnify the small portion of the graph inside the square, we would see a picture very similar to the original graph.

For more about the interesting behavior of (1) and about the dynamics of iterations of simple maps, see *Chaos and Fractals,* edited by Robert Devaney and Linda Keen and published by the American Mathematical Society, ISBN:0-8218-0137-6.

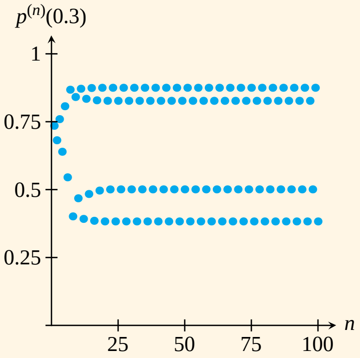

FIGURE 1.84 When $c = 3.5$, the list of compositions soon enters a cycle of length four.

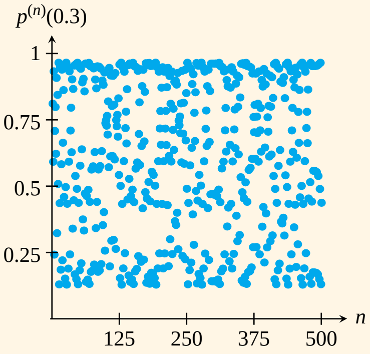

FIGURE 1.85 When $c = 3.862$, the list of compositions does not appear to enter a cycle.

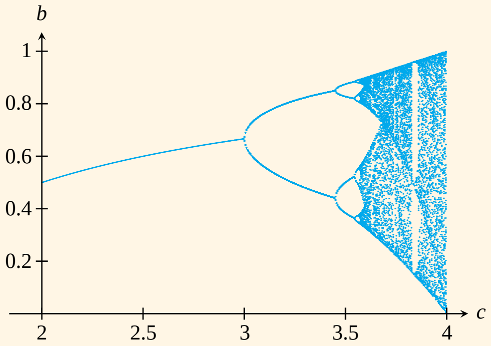

FIGURE 1.86. The graph indicates the cycle values for some values of c.