1.7 The Derivative

In his autobiography *Surely You're Joking, Mr. Feynman!* (Norton Publishing, 1985), physicist Richard Feynman tells the story of how he once pitted his pencil-and-paper arithmetic abilities against the skills of a merchant with an abacus.

They started with an addition problem, and the merchant was first with the answer. Next came a problem in multiplication. Again the merchant gave the answer first, but just barely ahead of Feynman. The next round tested division skills, and the result was a tie. For the last problem, the merchant suggested cube roots. Finding cube roots on an abacus is very difficult, but the merchant was evidently proud of his ability to do so. The party supplying the numbers challenged the competitors to find the cube root of 1729.03. Onlookers were amazed to see Feynman think a moment, then write the number 12.002 well before the merchant gave his answer of 12.

Feynman writes that the number 1729.03 was lucky for him. As a physicist he knew that $12^3 = 1728$, because this gives the number of cubic inches in a cubic foot. So $\sqrt[3]{1729.03}$ must be a little larger than 12... but how much larger? Feynman knew from calculus that for real numbers x close to 12^3,

$$\sqrt[3]{x} \approx 12 + \frac{1}{432}(x - 1728). \tag{1}$$

With $x = 1729.03$, this gave $\sqrt[3]{1729.03} \approx 12 + 1.03/432$. Feynman was able to quickly do the mental division to see that $1/432 \approx 0.002$. Let's try Feynman's approximation (1) for a few other numbers. Letting $c(x) = 12 + \dfrac{1}{432}(x - 1728)$ we find

$$\begin{array}{lcl}
c(1728) = 12 & \text{and} & \sqrt[3]{1728} = 12 \\
c(1725) \approx 11.9931 & \text{and} & \sqrt[3]{1725} \approx 11.9931 \\
c(1740) \approx 12.0278 & \text{and} & \sqrt[3]{1740} \approx 12.0277 \\
c(5000) \approx 19.5741 & \text{and} & \sqrt[3]{5000} \approx 17.0998
\end{array}$$

The first three examples support Feynman's statement that $c(x)$ is close to $\sqrt[3]{x}$ when x is close to 1728. On the other hand, the last entry shows that $c(x)$ may not be close to $\sqrt[3]{x}$ for *all* values of x. These numerical examples raise many interesting questions. Why should $c(x)$ be close to $\sqrt[3]{x}$ for any values of x close to 12^3? Suppose that we want $\sqrt[3]{x}$ accurate to three decimal places. For what values of x can we use $c(x)$? For x not near 12^3, is there another approximation that we can use, and if so, how do we find it? We can answer many of these questions after we take a closer look at the rate of change. (See also Problem 38.)

The Derivative

In Sections 1.3, 1.4, and 1.5 we discussed the rate of change of the pressure in a piston chamber with respect to volume, the rate of change of temperature in an autoclave with respect to time, and the rate of change of a population with respect to time. These rates of change are related in that they ask how a variable y changes with respect to a variable x, given that x and y are related by an equation of the form $y = f(x)$. We now define the *derivative* of a function f at a point $x = a$ of its domain.

DEFINITION Derivative

Let f be a function and suppose that f is defined at the point a (i.e., $f(a)$ is defined). The **derivative of f at a** is the number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{2}$$

provided that this limit exists. If this limit does not exist, we say f has no derivative at $x = a$.

The right side of (2) is identical to the limit expression used to describe the rate of change algorithm in Section 1.5. If we are in the context in which a variable y is related to a variable x through an equation $y = f(x)$, then the rate of change of y with respect to x when $x = a$ is $f'(a)$, the derivative of f at a.

If we are in a context in which our interest is primarily in the graph of $y = f(x)$ and, in particular, the slope of the "line" we see when we zoom in on the graph of f near $(a, f(a))$, then we may interpret the quotient

$$\frac{f(a+h) - f(a)}{h}$$

as the slope of the line joining the points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph of $y = f(x)$. See Fig. 1.70. We expect that as $h \to 0$, the slope of this line approaches the slope of the graph of f at a.

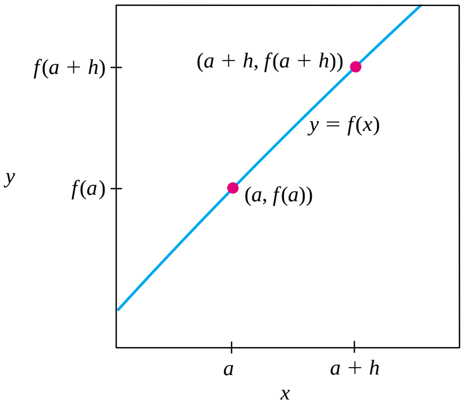


FIGURE 1.70 We see a "line" of slope $f'(a)$ when we zoom in on the graph of $y = f(x)$ near $(a, f(a))$.

Notation for the Derivative

There are many different notations for the derivative (or rate of change). The notation used often depends on how the function f is presented or on how the derivative is to be used. When a function is defined by a formula $y = f(x)$, the following are all notations for the derivative of f (or y) with respect to x at $x = a$:

$$f'(a), \quad \left.\frac{dy}{dx}\right|_{x=a}, \quad \left.\frac{df}{dx}\right|_{x=a}, \quad D_x f(a).$$

We shall use the first three of these notations.

The Derivative as a Function

If x is equal to a specific number, say $x = a$, then the derivative of f at a is the number denoted by $f'(a)$. If x is variable, we can think of the values $f'(x)$ as defining a function f'.

The Derivative as a Function

Let f be a function. The derivative function f' is defined by the equation

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \tag{3}$$

and is defined for all x for which this limit exists.

The derivative of f at a, $f'(a)$, is the value of the function f' at the value $x = a$. When $y = f(x)$, we will also use

$$y', \quad \frac{dy}{dx}, \quad \frac{df}{dx}, \quad \text{and} \quad \frac{d}{dx} f(x)$$

to denote the function f'. The $\frac{d}{dx}$ notation was introduced by Gottfried Liebnitz who, along with Isaac Newton, shares credit for the invention of calculus. The symbol $\frac{d}{dx}$ is often called an *operator;* it operates on a function f to give the derivative of f. Thus we can write

$$\frac{d}{dx} f(x) = f'(x).$$

EXAMPLE 1 Let $y = f(x) = x^2$. Find

a) $f'(1)$ **b)** $f'(x)$.

Solution

a) Setting $a = 1$ in (2) we have

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h\to 0} \frac{(1 + 2h + h^2) - 1}{h} = \lim_{h\to 0}(2 + h) = 2. \end{aligned}$$

b) Using (3),

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} = \lim_{h\to 0}(2x + h) = 2x. \end{aligned}$$

Thus $f'(x) = 2x$. Setting $x = 1$ in this result we find $f'(1) = 2 \cdot 1 = 2$, as found in part **a)**.

Δx and Δy Notation

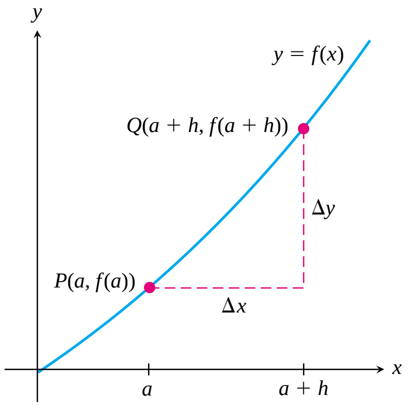


FIGURE 1.71 $Q(a+h, f(a+h))$ is near $P(a, f(a))$ on the graph of $y = f(x)$.

In Sections 1.3 and 1.4 we estimated the rate of change of a function f at a point $x = a$ by calculating the slope of the line determined by the point $P = (a, f(a))$ and a nearby point $Q = (a+h, f(a+h))$ on the graph of $y = f(x)$. Because Q was close to P, we knew that h had to be a small number. (See Fig. 1.71.) The slope of the line through P and Q was found by first calculating

$$\Delta y = \text{change in } y = f(a+h) - f(a)$$

and

$$\Delta x = \text{change in } x = (a+h) - a = h,$$

and then forming the quotient

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

In view of the definition of *derivative*, the rate of change of f with respect to x at $x = a$ is then

$$\left.\frac{df}{dx}\right|_{x=a} = f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}. \tag{4}$$

Notations such as Δy and Δx are often used to represent small changes in quantities. This notation reminds us that the rate of change (or derivative) can be estimated by calculating the quotient $\dfrac{\Delta y}{\Delta x}$ for small Δx. This quotient also looks very much like the Leibnitz notation, $\dfrac{dy}{dx}$, for the derivative. The Leibnitz notation reminds us that the derivative is a limit of a ratio of small quantities.

Another Formula for the Derivative

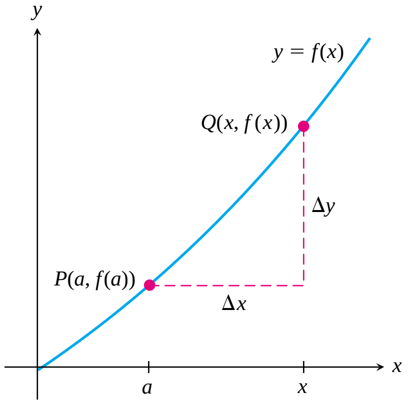


FIGURE 1.72 $Q(x, f(x))$ is near $P(a, f(a))$ on the graph of $y = f(x)$.

In the preceding discussion we used $Q = (a+h, f(a+h))$ to denote a point close to $P = (a, f(a))$. We could just as easily have denoted this nearby point $(x, f(x))$ with the understanding that x is close to a. This is illustrated in Fig. 1.72. Substituting x for $a + h$ in (2) and noting that $h \to 0$ is equivalent to $x \to a$ gives us another form for the definition of the derivative of f at a,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{5}$$

Either (2) or (5) can be used to compute a derivative. Depending on the function f, calculations with the two formulas can involve different amounts of algebra.

Interpreting the Derivative

To effectively use the derivative as a tool, it is important to know how to interpret it in different contexts. Remember that the derivative (or rate of change) of $y = f(x)$ carries the units

$$\frac{\text{units of } y}{\text{units of } x}.$$

Knowing these units often helps in determining the meaning of the derivative.

EXAMPLE 2 A high diver jumps from a tower into a pool of water 100 feet below. After she has fallen for t seconds, her height (in feet) above the pool is $H(t) = 100 - 16t^2$. Find the derivative of H and discuss the meaning of the derivative. How long after she jumps does the diver reach the pool? What is the derivative at this time, and what does it tell us?

Solution

From (3), the derivative is

$$\begin{aligned} H'(t) &= \lim_{h\to 0} \frac{H(t+h) - H(t)}{h} \\ &= \lim_{h\to 0} \frac{(100 - 16(t+h)^2) - (100 - 16t^2)}{h}. \end{aligned}$$

We cannot evaluate this limit by simply substituting $h = 0$ because this leads to $0/0$. Instead we do some algebra to find an equivalent expression whose limit we can determine. After simplifying the numerator in the last expression, we can factor and cancel an h from the numerator and denominator to get

$$H'(t) = \lim_{h\to 0} \frac{-32th - 16h^2}{h} = \lim_{h\to 0}(-32t - 16h) = -32t.$$

Since the height $H(t)$ is in feet and time t is in seconds, the units for the derivative are feet/second. Thus at time t the rate of change of the diver's height with respect to time is $-32t$ ft/s. Because the units are units of velocity, the derivative tells us how fast the diver is moving at time t. The minus sign tells us that when we zoom in on the graph of $H = H(t)$ near a point $(t, H(t))$, we will see a "line" of negative slope. Hence $H(t)$ decreases as time t increases. (This makes sense because as the diver falls, her height above the water is decreasing.)

When the diver reaches the water, her height $H(t)$ above the pool will be 0. Solving

$$H(t) = 100 - 16t^2 = 0 \quad \text{we find} \quad t = \pm 2.5 \text{ seconds.}$$

Because the t we seek must be positive, we conclude that the diver hits the water 2.5 seconds after jumping off the tower. See Fig. 1.73. The derivative at this time is

$$H'(2.5) = -32(2.5) = -80 \text{ ft/s}.$$

Thus the diver is moving downward at 80 ft/s when she hits the water.

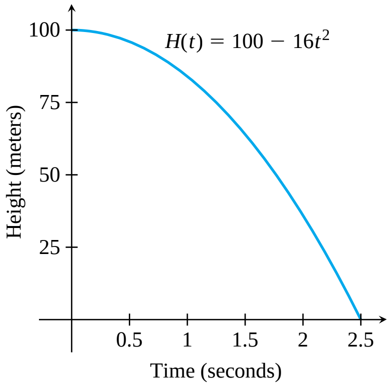


FIGURE 1.73 The graph illustrates the height of the diver as a function of time.

EXAMPLE 3 According to Newton's law of gravitation, a 100-kg object r meters above the surface of the Earth is subject to a gravitational force of

$$F(r) = \frac{100GM_e}{(R_e + r)^2} \text{ newtons (N)},$$

where R_e is the radius of the Earth ($\approx 6.37 \times 10^6$ m), M_e is the mass of the Earth ($\approx 5.98 \times 10^{24}$ kg), and G is the gravitational constant ($\approx 6.67 \times 10^{-11}$ Nm^2/kg^2). Find the rate of change of F with respect to r as the object moves away from the surface of the Earth along a ray from the center of the Earth. What is the rate of change when the object is 10^7 m above the surface of the Earth?

Solution

The rate of change is the derivative. Using (5), we have

$$F'(r) = \lim_{t\to r} \frac{F(t) - F(r)}{t - r} = \lim_{t\to r} \frac{\dfrac{100GM_e}{(R_e + t)^2} - \dfrac{100GM_e}{(R_e + r)^2}}{t - r}.$$

Adding the fractions in the numerator of this expression gives

$$F'(r) = \lim_{t\to r} \frac{100GM_e((R_e + r)^2 - (R_e + t)^2)}{(R_e + r)^2(R_e + t)^2(t - r)}.$$

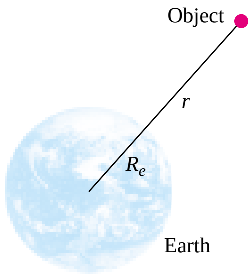
Next factor $(R_e + r)^2 - (R_e + t)^2$ as a difference of squares to obtain

$$\begin{aligned} F'(r) &= \lim_{t\to r} \frac{100GM_e(r - t)(2R_e + r + t)}{(R_e + r)^2(R_e + t)^2(t - r)} \\ &= \lim_{t\to r} \frac{-100GM_e(2R_e + r + t)}{(R_e + r)^2(R_e + t)^2} \\ &= \frac{-100GM_e(2R_e + 2r)}{(R_e + r)^4} \\ &= \frac{-200GM_e}{(R_e + r)^3}. \end{aligned}$$

The units for the derivative are newtons/meter. When $r = 10^7$ m, the derivative has value

$$F'(10^7) = \frac{-200GM_e}{(R_e + 10^7)^3} \approx -1.82 \times 10^{-5}\ \text{N/m}.$$

When the object is 10^7 m above the Earth's surface, the gravitational force on the object is $F(10^7) \approx 148.8$ N. See Fig. 1.74. The derivative tells us that the force on the object decreases by about 1.82×10^{-5} N if the object moves 1 m farther from the surface of the Earth.



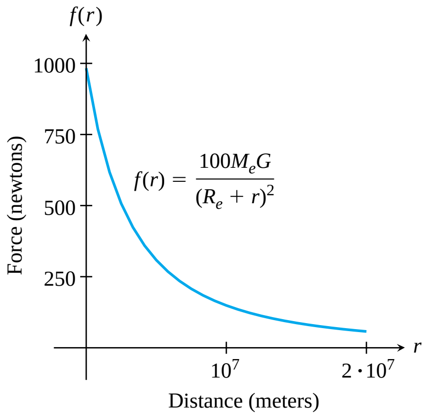


FIGURE 1.74 The graph illustrates the force on the object as a function of the distance from the Earth's surface.

EXAMPLE 4 The owner of the Good Lookin' Glass Company asks a group of engineers and accountants to reflect on the cost of manufacturing several one-way mirrors. They report that the cost of producing $N > 0$ mirrors will be about

$$C(N) = 3000 + 5N + 2\sqrt{N} \quad \text{dollars.} \tag{6}$$

Compute the derivative of the function C and discuss the significance of the derivative.

Solution

We first discuss the meaning of (6). The glass company cannot manufacture $100\frac{1}{2}$ or $\sqrt{2}$ mirrors, but only a positive whole number of mirrors. Thus $C(N)$ is only defined for $N = 1, 2, 3, \ldots$. Thus, the graph of $C(N)$ shows just the individual points $(1, C(1))$, $(2, C(2))$, $(3, C(3)), \ldots$ (see Fig. 1.75). Such a function does not have a derivative. When we zoom in on this graph, we will never see a straight line. This suggests that the rate of change (or derivative) of $C(N)$ with respect to N does not exist. Indeed, if the derivative of C did exist, it would be given by

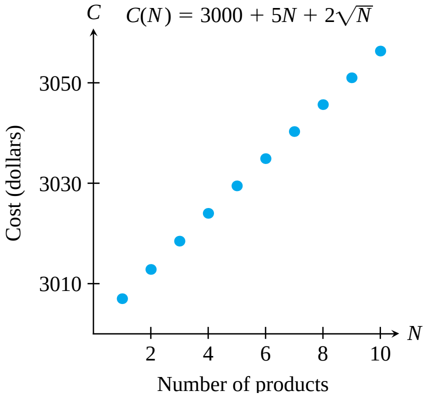


FIGURE 1.75 The cost function is defined only for nonnegative integers N.

$$\lim_{t\to N} \frac{C(t) - C(N)}{t - N}. \tag{7}$$

However, the quotient $(C(t) - C(N))/(t - N)$ is not defined for t in intervals on either side of N because any such interval contains noninteger values of t. This means that the conditions for the existence of the limit (7) are not satisfied. Hence the limit (7) does not exist.

Even though (6) may not be meaningful from a manufacturing point of view for nonintegers N, useful information can be obtained by assuming $C(N)$ is defined for all positive N. We can then compute $C'(N)$ for any positive N:

$$\begin{aligned} C'(N) &= \lim_{t\to N} \frac{C(t) - C(N)}{t - N} \\ &= \lim_{t\to N} \frac{\left(3000 + 5t + 2\sqrt{t}\right) - \left(3000 + 5N + 2\sqrt{N}\right)}{t - N} \\ &= \lim_{t\to N} \left(\frac{5(t - N)}{t - N} + \frac{2\left(\sqrt{t} - \sqrt{N}\right)}{\left(\sqrt{t} - \sqrt{N}\right)\left(\sqrt{t} + \sqrt{N}\right)} \right) \\ &= \lim_{t\to N} \left(5 + \frac{2}{\sqrt{t} + \sqrt{N}} \right) \\ &= 5 + \frac{1}{\sqrt{N}}. \end{aligned}$$

When $C(N)$ is the cost of manufacturing N units of a product, $C'(N)$ has units of dollars/product and is called the **marginal cost.** The marginal cost can be used as an estimate of the cost of producing a unit of the product after N units have already been manufactured. That is, $C'(N)$ is used as an estimate of $C(N + 1) - C(N)$. To see that this is reasonable, note that

$$C'(N) = \lim_{h\to 0} \frac{C(N + h) - C(N)}{h}.$$

Assuming that $h = 1$ is "small" in this context, we have

$$C'(N) \approx \frac{C(N + 1) - C(N)}{1} = C(N + 1) - C(N).$$

In manufacturing processes, where the average cost per product goes down as more products are produced, this approximation is often a good one. To check this, assume that 1000 mirrors have been produced. The extra cost to produce mirror 1001 is

$$\begin{aligned} C(1001) - C(1000) &= \left(3000 + 5 \cdot 1001 + 2\sqrt{1001}\right) \\ &\quad - \left(3000 + 5 \cdot 1000 + 2\sqrt{1000}\right) \\ &\approx \$5.031615. \end{aligned}$$

The approximation to this value given by the marginal cost is

$$C(1001) - C(1000) \approx C'(1000) = 5 + 1/\sqrt{1000} \approx \$5.031623.$$

Pretty close!

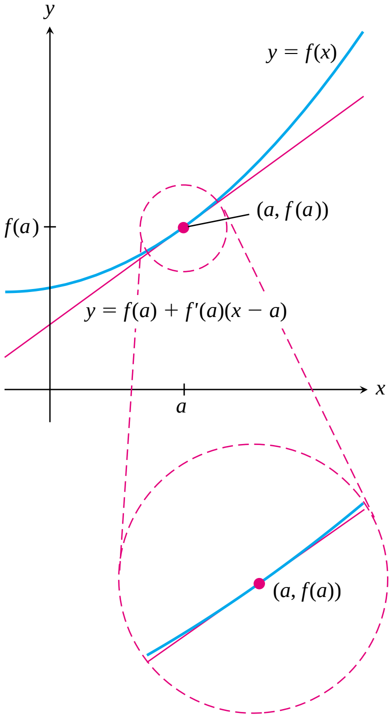


FIGURE 1.76 The graph of the tangent line is close to the graph of the function near the point of tangency.

The Tangent Line

Let f be a function with derivative $f'(a)$ at $x = a$. When we zoom in on the graph of $y = f(x)$ near the point $(a, f(a))$, the graph appears to be a straight line with slope $f'(a)$. See Fig. 1.76. This motivates the following definition.

DEFINITION Tangent Line

Let f have derivative $f'(a)$ at $x = a$. The line with slope $f'(a)$ through $(a, f(a))$ is the **tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$.

Because the line tangent to $y = f(x)$ at the point $(a, f(a))$ has slope $f'(a)$, we can easily write down an equation for the tangent line:

$$y - f(a) = f'(a)(x - a)$$

or

$$y = f(a) + f'(a)(x - a).$$

EXAMPLE 5 Find the equation of the line tangent to the graph of $y = x^2$ at the point $(-2, 4)$. On the same set of axes, sketch the graph of $y = x^2$ and the graph of the tangent line.

Solution

Let $f(x) = x^2$. In Example 1 we showed that $f'(x) = 2x$. Thus the desired tangent line has slope $f'(-2) = 2(-2) = -4$ and contains the point $(-2, 4)$. The equation for this line is

$$y - 4 = (-4)(x - (-2)).$$

This simplifies to $y = -4x - 4$. The graphs of $y = x^2$ and the tangent line are shown in Fig. 1.77.

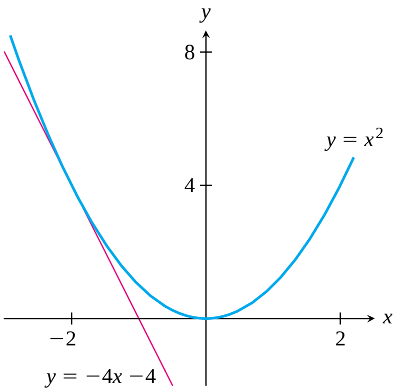


FIGURE 1.77 The line tangent to the graph of $y = x^2$ at the point $(-2, 4)$ has slope -4.

The Tangent Line as an Approximation If we draw the line tangent to the graph of $y = f(x)$ at the point $(a, f(a))$, then zoom in on the graph near this point, the graph and the tangent line will be almost indistinguishable. This is illustrated in Fig. 1.76.

If the graphs of the functions

$$y = f(x) \quad \text{and} \quad y = f(a) + f'(a)(x - a)$$

are almost indistinguishable, then the values of these two expressions must be close. Thus for x close to a,

$$f(x) \approx f(a) + f'(a)(x - a). \tag{8}$$

Expression (8) says that when x is close to a, the function value $f(x)$ is approximated by the corresponding y-value for the tangent line.

Java Applet
Tangent Lines
Explores the connection between the slope of a line tangent to the graph of a function and the function's derivative. Students can graph a function $f(x)$ and the line tangent to the graph of $y = f(x)$ for a given x value.

EXAMPLE 6

Find the equation of the line tangent to

$$f(x) = 1/(x+1)$$

at $(2, 1/3)$ and use this equation to approximate $f(1.93) = 1/(1.93+1)$.

Solution

The tangent line passes through the point of tangency, $(2, 1/3)$. The slope of the tangent line is the derivative of f at $x = 2$,

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{\dfrac{1}{x+1} - \dfrac{1}{3}}{x - 2}.$$

Add the fractions in the numerator and simplify the resulting complex fraction. We then have

$$f'(2) = \lim_{x\to 2} \frac{2 - x}{3(x+1)(x-2)} = \lim_{x\to 2} \frac{-1}{3(x+1)} = -\frac{1}{9}.$$

The equation of the line tangent to $y = 1/(x+1)$ at $(2, 1/3)$ is

$$y = f(2) + f'(2)(x-2) = \frac{1}{3} - \frac{1}{9}(x-2).$$

When x is close to 2, the y coordinates of the graph of $y = 1/(x+1)$ and the tangent line should be close. See Fig. 1.78. Hence for x near 2,

$$\frac{1}{x+1} \approx \frac{1}{3} - \frac{1}{9}(x-2).$$

Substitute $x = 1.93$ into this expression to get

$$f(1.93) = \frac{1}{1.93+1} \approx \frac{1}{3} - \frac{1}{9}(1.93-2) \approx 0.341111.$$

How does this compare with the value of $(1.93+1)^{-1}$ given by your calculator?

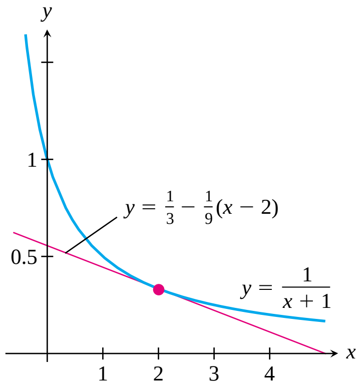


FIGURE 1.78 The line tangent to $y = 1/(x+1)$ at $(2, 1/3)$.

Error in the Tangent Line Approximation We can use the techniques developed in Section 1.6 to estimate the error in the tangent line approximation. Let

$$\frac{f(x) - f(a)}{x - a} - f'(a) = E(x). \tag{9}$$

Because

$$\lim_{x\to a} E(x) = \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a} - f'(a)\right) = f'(a) - f'(a) = 0,$$

we can make $E(x)$ small by taking x close to a. Solving (9) for $f(x)$, we find

$$f(x) = f(a) + f'(a)(x - a) + E(x)(x - a). \tag{10}$$

Hence the difference between the function value $f(x)$ and the value $f(a) + f'(a)(x - a)$ of the tangent line expression is

$$f(x) - (f(a) + f'(a)(x - a)) = E(x)(x - a). \tag{11}$$

From (11) we see that this difference is small compared to $|x - a|$ when $E(x)$ is small, that is, when x is close to a. The error is illustrated in Fig. 1.79.

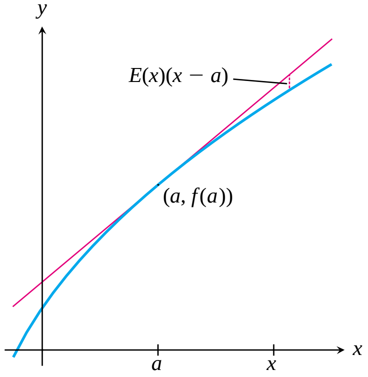


FIGURE 1.79 The difference between $f(x)$ and the y-coordinate of the tangent line is $E(x)(x - a)$.

EXAMPLE 7 The line tangent to $y = \sin x$ at $(0, 0)$ is used to approximate $\sin x$ for $-0.5 < x < 0.5$. Discuss the error in the approximation.

Solution

We first find the tangent line. Using (2) with $f(x) = \sin x$ and $a = 0$, the slope of the tangent line is

$$\lim_{h\to 0} \frac{\sin h - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1.$$

(The value of this limit was found in Example 8 of Section 1.5.) Because the tangent line has slope 1 and contains the point $(0, 0)$, its equation is $y = x$.

We now need to discuss the error in the approximation

$$\sin x \approx x$$

for $-0.5 \le x \le 0.5$. In Example 5 of Section 1.6 we showed that

$$|\sin x - x| \le 0.05|x| \le 0.025$$

for such x. Thus if we use the approximation $\sin x \approx x$ for x in the interval $(-0.5, 0.5)$, the error is no bigger than 0.025 in absolute value. Alternatively, the error is no bigger than 5% of $|x|$. The graph of $y = \sin x$ and the tangent line are shown in Fig. 1.80.

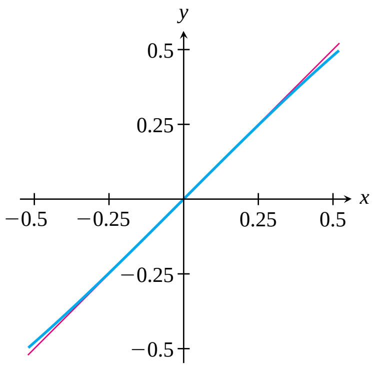


FIGURE 1.80 The line $y = x$ is tangent to the graph of $y = \sin x$ at $(0, 0)$.

EXAMPLE 8 The line tangent to $y = \sqrt{x}$ at $(1, 1)$ is used to approximate $\sqrt{x}$ for $0.9 < x < 1.1$. Discuss the error in the approximation.

Solution

The slope of the tangent line is the derivative of $\sqrt{x}$ at $x = 1$. Using (5) with $f(x) = \sqrt{x}$ and $a = 1$ the derivative is

$$\lim_{x\to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x\to 1} \frac{\sqrt{x} - 1}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \frac{1}{2}.$$

The tangent line is the line of slope $1/2$ that contains the point $(1, 1)$. This line has equation

$$y = \frac{1}{2}x + \frac{1}{2}.$$

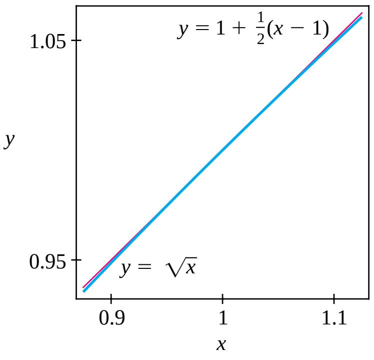


FIGURE 1.81 The line $y = \frac{1}{2}x + \frac{1}{2}$ is tangent to the graph of $y = \sqrt{x}$ at $(1, 1)$.

We now need to consider the error in the approximation

$$\sqrt{x} \approx \frac{1}{2}x + \frac{1}{2} \tag{12}$$

for $0.9 < x < 1.1$. In Example 6 of Section 1.6 we showed that

$$\left|\sqrt{x} - \left(\frac{1}{2}x + \frac{1}{2}\right)\right| \le 0.015|x - 1| \le (0.015)(0.01) = 0.0015$$

for such x. Thus if we use the approximation (12) for $0.9 < x < 1.1$, the error is no more than 0.0015 in absolute value. In addition, the error is no bigger than 1.5% of the value of $|x - 1|$. The graph of $y = \sqrt{x}$ and the tangent line are shown in Fig. 1.81.

Derivatives and Continuous Functions As another consequence of (10), we obtain a necessary condition for a function f to have a derivative at a point $x = a$.

Continuity of Differentiable Functions

If the function f has a derivative at $x = a$, then f is continuous at $x = a$; that is,

$$\lim_{x\to a} f(x) = f(a).$$

To justify this statement, take the limit as $x \to a$ of both sides of (10):

$$\lim_{x\to a} f(x) = \lim_{x\to a}(f(a) + f'(a)(x - a) + E(x)(x - a)).$$

Applying results from Section 1.5 for limits of sums and products of functions, we obtain

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} f(a) + \lim_{x\to a}(f'(a)(x - a)) + \lim_{x\to a}(E(x)(x - a)) \\ &= f(a) + f'(a)\lim_{x\to a}(x - a) + \left(\lim_{x\to a} E(x)\right)\left(\lim_{x\to a}(x - a)\right) \\ &= f(a) + f'(a)\cdot 0 + 0\cdot 0 \\ &= f(a).\end{aligned}$$

Because $\lim_{x\to a} f(x) = f(a)$, the function f is continuous at $x = a$.

Exercises 1.7

Exercises 1–10: Determine the derivative in each case. Use (2) for some of the problems and (5) for others.

1. $f(x) = x^2 - 3$

2. $y = 2x^2 - 6$

3. $h(t) = -3t^2 + 4t - \sqrt{2}$

4. $g(u) = \dfrac{2}{u + 2}$

5. $y = \dfrac{4}{2 - x}$

6. $s(t) = t - 2\sqrt{t}$

7. $y = 2t + 3\sqrt{t+9}$

8. $r(s) = as^2 + bs + c$ where a, b, c are constants.

9. $y = \dfrac{1}{ax+b}$ where a and b are constants.

10. $h(t) = \sqrt{at+b}$ where a and b are constants.

Exercises 11–17: Find an equation for the line tangent to the graph of the function at the specified point.

11. $y = x^2 - 3x$ at $(1, -2)$

12. $s(t) = -3t^2 - 2t + 1$ at $(-2, -7)$

13. $h(r) = 4r + 2$ at $(1, 6)$

14. $y = \dfrac{1}{x}$ at $(a, 1/a)$, $a \neq 0$

15. $g(u) = \dfrac{2}{u^2 - 1}$ at $(2, 2/3)$

16. $w = \sqrt{2v - 1}$ at $\left(b, \sqrt{2b-1}\right)$, $b > 1/2$

17. $f(x) = 2|x|$ at $(2, 4)$

18. Use the equation for the line tangent to $y = x^2$ at $(4, 16)$ to approximate $(4.14)^2$ and $(3.91)^2$. In each case, compute the error in the approximation.

19. Use the equation for the line tangent to $y = 1/x$ at $(3, 1/3)$ to approximate $1/(3.15)$ and $1/(2.85)$. In each case, compute the error in the approximation.

20. In Chapter 2 we will see that the derivative of $\sin x$ is $\cos x$.

- **a.** Find an equation for the line tangent to the graph of $y = \sin x$ at $\left(\pi/3, \sqrt{3}/2\right)$.
- **b.** Use the result of part a to find an approximation to $\sin(\pi/3 + 0.1)$. (All angle measures are in radians.) Use your calculator to check the accuracy of the approximation.

21. In Chapter 2 we will see that the derivative of $\tan x$ is $\sec^2 x$.

- **a.** Find an equation for the line tangent to the graph of $y = \tan x$ at $(\pi/4, 1)$.
- **b.** Use the result of part a to find an approximation to $\tan(\pi/4 - 0.05)$. (All angle measures are in radians.) Use your calculator to check the accuracy of the approximation.

22. The planet Jupiter has mass $M \approx 1.9 \times 10^{27}$ kg and radius $R \approx 6.98 \times 10^{7}$ m. Suppose the 100-kg mass of Example 3 is 10^7 m above the surface of Jupiter. Find the rate of change of the gravitational force as the object moves away from Jupiter along a ray through the center of the planet. Discuss the meaning of this rate of change.

23. Repeat Exercise 22 for the moon. The moon has mass $M \approx 7.34 \times 10^{22}$ kg and radius $R \approx 1.74 \times 10^{6}$ m.

24. The accountants at the Seed & Sod Turf Company estimate that the cost of a grassroots advertising campaign that will reach $N \times 10^5$ consumers is roughly $C(N) = 5000 + 1000\sqrt{N} + 0.05(N-1)^2$ dollars.

- **a.** Find the derivative of $C(N)$ and discuss the meaning of this derivative as a marginal cost.
- **b.** Use the derivative to estimate the extra cost of a campaign to reach a total of 1,100,000 consumers over that of a campaign to reach 1,000,000 consumers.

25. A function f has derivative

$$f'(x) = \frac{1}{x^2 + 1}.$$

Given that $f(0) = 2$:

- **a.** Find an equation for the line tangent to the graph of $y = f(x)$ at the point $(0, 2)$.
- **b.** Find approximations to $f(0.05)$ and $f(-0.1)$.

26. A function h has derivative

$$h'(u) = u \sin(2u).$$

Given that $h(3\pi/4) = -3$, find an approximation to

$$h\left(\frac{3\pi}{4} + 0.15\right).$$

27. The line $y = 5x - 4$ is tangent to the graph of $y = g(x)$ at the point $(-1, -9)$. Find $g'(-1)$ and an approximation to $g(-0.88)$.

28. The line $y = -2t + 3$ is tangent to the graph of $y = h(t)$ at the point $(3, -3)$. Find $h'(3)$ and an approximation to $h(3.1)$.

29. Let $f(x) = x^3$.

- **a.** Find an equation for the line tangent to the graph of $y = f(x)$ at the point $(-2, -8)$.
- Ⓣ **b.** Let $y = t(x)$ be an equation for the tangent line of part **a**. Suppose we wish to use the approximation

$$f(x) \approx t(x)$$

for x values near -2. Find r so that when $|x - (-2)| < r$, the error in the approximation is

$$|f(x) - t(x)| < 0.01|x - (-2)|.$$

(See Example 6 in Section 1.6.)

30. Let $f(x) = 1/\sqrt{x+1}$.

- **a.** Find an equation for the line tangent to the graph of $y = f(x)$ at the point $(3, 1/2)$.
- Ⓣ **b.** Let $y = t(x)$ be an equation for the tangent line of part **a**. Suppose we wish to use the approximation

$$f(x) \approx t(x)$$

for x near 3. Find r so that when $|x - 3| < r$, the error in the approximation is

$$|f(x) - t(x)| < 0.01|x - 3|.$$

(See Example 6 in Section 1.6.)

31. Let f be a function with $f(4) = 3$ and $f'(4) = 7$. Define $h(x) = f(-x)$.

- **a.** Tell why the point $(-4, 3)$ is on the graph of $y = h(x)$.
- **b.** What is the value of $h'(-4)$? Justify your answer by comparing the graphs of $y = f(x)$ and $y = h(x) = f(-x)$.
- **c.** By arguing graphically, tell why if the derivative of $h(x)$ at $x = a$ is defined, then $h'(a) = -f'(-a)$.

32. Let f be a function with $f(4) = 3$ and $f'(4) = 7$. Define $h(x) = -f(x)$.

- **a.** Tell why the point $(4, -3)$ is on the graph of $y = h(x)$.
- **b.** What is the value of $h'(4)$? Justify your answer by comparing the graphs of $y = f(x)$ and $y = h(x) = -f(x)$.
- **c.** By arguing graphically, tell why if the derivative of $h(x)$ at $x = a$ is defined, then $h'(a) = -f'(a)$.

33. Let f be a function with $f(4) = 3$ and $f'(4) = 7$. Define $h(x) = 8f(x)$.

- **a.** Tell why the point $(4, 24)$ is on the graph of $y = h(x)$.
- **b.** What is the value of $h'(4)$? Justify your answer by comparing the graphs of $y = f(x)$ and $y = h(x) = 8f(x)$.
- **c.** Express $h'(a)$ in terms of the derivative of f. Justify your answer with the aid of graphs.

34. Let f be a function with $f(4) = 3$ and $f'(4) = 7$. Define $h(x) = f(x + 7)$.

- **a.** Tell why the point $(-3, 3)$ is on the graph of $y = h(x)$.
- **b.** What is the value of $h'(-3)$? Justify your answer by comparing the graphs of $y = f(x)$ and $y = h(x) = f(x + 7)$.
- **c.** Express $h'(a)$ in terms of the derivative of f. Justify your answer with the aid of graphs.

35. The limit statement

$$\lim_{x \to 2} \frac{(x^9 - 4x^7 + 3x - 2) - 4}{x - 2} = 515$$

is a statement about the derivative of some function f at some value $x = a$. What are f, a, and $f'(a)$?

36. The limit statement

$$\lim_{h \to 0} \frac{\sec(\pi/4 + h) - \sqrt{2}}{h} = \sqrt{2}$$

is a statement about the derivative of some function g at some $t = a$. What are g, a, and $g'(a)$?

37. A function f satisfies

$$f(1) = f(2) = f(3) = f(4) = f(5) = 0,$$
$$f'(1) = f'(3) = f'(5) = 1,$$

and

$$f'(2) = f'(4) = -1.$$

Use this information to sketch a possible graph for $y = f(x)$.

38. In this problem we discuss Richard Feynman's approximation (1) to the cube root function.

- **a.** Show that

$$\lim_{x \to 1728} \frac{\sqrt[3]{x} - 12}{x - 1728} = \frac{1}{432}.$$

It will be helpful to keep in mind that $12^3 = 1728$. The factorization

$$b^3 - a^3 = (b - a)(b^2 + ba + a^2)$$

may also prove useful.

- **b.** Use the result of part a to obtain the approximation (1).
- T **c.** Use graphical means to investigate the error in the approximation for $1720 \leq x \leq 1736$.

Review of Key Concepts

We began this chapter with a review of functions. We saw that functions can be represented in a number of ways: symbolically (with a formula), graphically, or numerically (with a table of values). Functions represented in these ways arise regularly in science, engineering, and mathematics. It is important to know how to work with functions in all of these forms—not only with paper and pencil, but also with the aid of a calculator or computer algebra system.

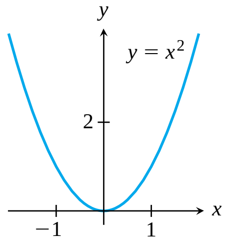
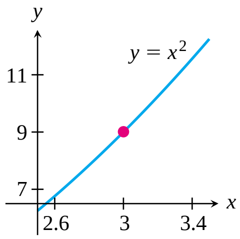
Following our review of functions, we interpreted the slope of a line as a rate of change and used this to motivate a definition for the rate of change of an arbitrary function. For functions whose graphs look like lines as we zoom in, we defined the rate of change as the slope of this line. From this graphical definition we developed a numerical understanding of the rate of change and an analytic (or symbolic) definition for the rate of change.

The rate of change definition led to the idea of limit. Once limit was defined, we reformulated our definition of rate of change in a more precise form. We also saw that limits are very closely related to approximation and error. Because approximation is a very important idea in modern science, engineering, and mathematics, we spent some time discussing approximations and errors.

We concluded the chapter by defining the derivative of a function and noted that the derivative is really just another name for the rate of change. As one geometric application of the derivative, we defined the tangent line and noted that it is the “line” we see when we zoom in on a point of a graph. We saw that tangent lines can be used to obtain a good approximation to a function.

In the Chapter Summary that follows, we summarize many of these ideas in table form. For each concept, we present a general definition and a specific example and support both with a graph.

Chapter Summary

<table>
<tr><th colspan="3">Representing Functions</th></tr>
<tr>
<td>A function can be represented by a graph:
 </td>
<td>A function f can be represented by an equation:
$$f(x) = x^2.$$</td>
<td>A function f can be represented by a table of values:
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−0.9</td><td>0.81</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>3/2</td><td>9/4</td></tr><tr><td>3</td><td>9</td></tr></table></td>
</tr>
<tr><th colspan="3">Rate of Change</th></tr>
<tr>
<td>The rate of change at $x = a$ is the slope of the “line” we see when we zoom in on the point $(a, f(a))$.
 </td>
<td>The rate of change of $y = f(x)$ with respect to x at $x = a$ is
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$</td>
<td>If $y = f(x) = x^2$, the rate of change of y with respect to x at $x = 3$ is
$$\lim_{h\to 0}\frac{(3+h)^2-3^2}{h} = 6.$$</td>
</tr>
</table>