# 1.6 More Work with Limits

In the preceding section, we gave a "working" definition for *limit:*

**DEFINITION** Limit of a Function

Let $g$ be a function and let $a$ and $L$ be real numbers. If we can make $g(x)$ as close to $L$ as we like by taking $x$ close to $a$, but not equal to $a$, then we say

$g(x)$ has limit $L$ as $x$ approaches $a$.

We denote this by

$$\lim_{x \to a} g(x) = L. \tag{1}$$

Equation (1) is read "the limit of $g(x)$ as $x$ approaches $a$ equals $L$."

For some work with limits, we need to be more precise about the phrase "close to" used in this definition. When we talk about the number $g(x)$ being close to the number $L$, it is natural to ask, "how close?" To answer this question, we look at

$$|g(x) - L|, \tag{2}$$

that is, the distance between $g(x)$ and $L$. If $g(x)$ is close to $L$, then the distance $|g(x) - L|$ between them should be small. We will measure how close $g(x)$ is to $L$ by measuring how small (2) is. We do this with a number. For example, to say that $g(x)$ is within 0.01 of $L$ is the same as saying that

$$|g(x) - L| \le 0.01.$$

The definition of *limit* says: We can make $g(x)$ as close to $L$ as we like by taking $x$ close to $a$. This means that we can make (2) as small as we like by making $|x - a|$ sufficiently small. We illustrate this with some examples.

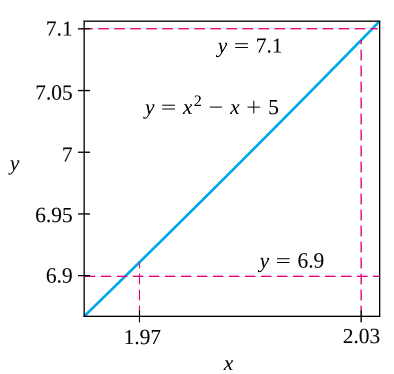


FIGURE 1.52 When $1.97 < x < 2.03$ we have $6.9 < x^2 - x + 5 < 7.1$.

**EXAMPLE 1** Let $g(x) = x^2 - x + 5$ and consider the statement

$$\lim_{x \to 2} g(x) = 7.$$

**a)** Show that if $|x - 2| < 0.03$, then $g(x)$ is within 0.1 of 7.

**b)** Suppose we want

$$|g(x) - 7| < 0.001. \tag{3}$$

Find a positive number $d$ so that (3) is true if $|x - 2| < d$.

**Solution**

**a)** We verify this graphically. First note that

$$|x - 2| < 0.03 \quad \text{is equivalent to} \quad 1.97 < x < 2.03.$$

The graph of $y = g(x)$ on this interval is shown in Fig. 1.52. We see that for $1.97 < x < 2.03$, the points on the graph lie between the lines

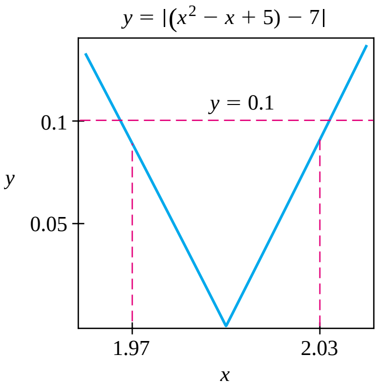


FIGURE 1.53 If $|x-2|<0.03$ then have $|(x^2-x+5)-7|<0.1$.

$y = 6.9$ and $y = 7.1$. This means that if

$$1.97 < x < 2.03 \quad \text{then} \quad 6.9 < g(x) < 7.1.$$

Hence if $|x-2| < 0.03$, then $g(x) = x^2 - x + 5$ is within 0.1 of 7.

For a different approach, graph $y = |g(x) - 7|$ for $1.97 < x < 2.03$. See Fig. 1.53. For $x$ in this interval, the points on the graph lie below the line $y = 0.1$. This means that

$$|g(x) - 7| < 0.1 \quad \text{if} \quad 1.97 < x < 2.03.$$

Thus $g(x) = x^2 - x + 5$ is within 0.1 of 7 when $x$ is within 0.03 of 2.

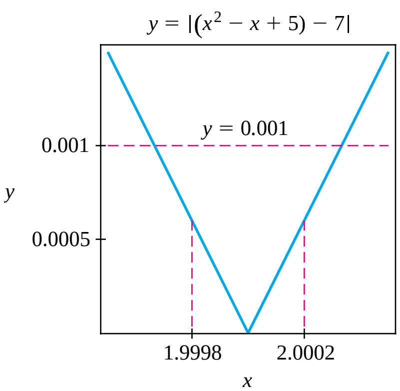


FIGURE 1.54 When $|x-2|<0.0002$ then $|(x^2-x+5)-7|<0.001$.

**b)** Because we want $|g(x) - 7| < 0.001$ for $x$ close to 2, draw the graph of $y = |g(x) - 7|$ on some small interval centered at 2. In Fig. 1.54 we see the graph for $1.9995 < x < 2.0005$. If we include the line $y = 0.001$ on the graph, we can then see that

$$y = |g(x) - 7| < 0.001 \quad \text{for} \quad 1.9998 < x < 2.0002,$$

that is, when

$$|x - 2| < 0.0002.$$

Hence we may take $d = 0.0002$.

For another solution, see Fig. 1.55. In this figure we show the graph of $y = g(x)$ for $|x - 2| < 0.0002$. Note that for $x$ values in this interval, the points of the graph lie between the lines $y = 6.999$ and $y = 7.001$. This again shows that 0.0002 is an acceptable value for $d$. Are there other acceptable values of $d$? How many other values? It is important to realize that graphs like those shown in Figs. 1.54 and 1.55 are usually not produced on the first attempt. The authors tried several graphs, with various parts of the domain and range specified, before producing pictures that show the needed details.

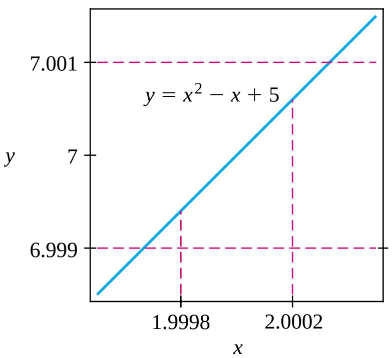


FIGURE 1.55 When $1.9998 < x < 2.0002$ we have $6.999 < g(x) < 7.001$.

**EXAMPLE 2** In the preceding section we showed that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Find $d > 0$ so that

$$\text{if } 0 < |\theta - 0| < d, \quad \text{then} \quad \left| \frac{\sin \theta}{\theta} - 1 \right| < 0.05. \tag{4}$$

## Solution

In Figure 1.56 we show the graph of $y = (\sin \theta)/\theta$ for $-1 < \theta < 1$, $\theta \neq 0$. Because we want

$$0.95 < \frac{\sin \theta}{\theta} < 1.05,$$

we include the lines $y = 0.95$ and $y = 1.05$ in the figure. The graph of $y = (\sin \theta)/\theta$ lies between these two horizontal lines when $-0.5 < \theta < 0$ or

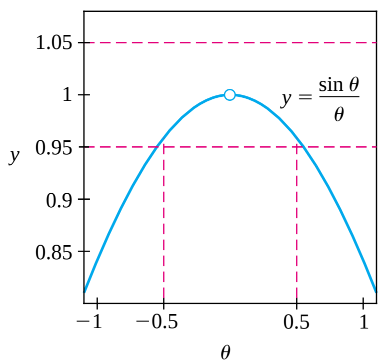


FIGURE 1.56 When $-0.5 < x < 0.5$ we have $0.95 < (\sin\theta)/\theta < 1.05$.

**TABLE 1.10**

| $\theta$ | $\frac{\sin\theta}{\theta}$ |
|---|---|
| −0.8 | 0.897 |
| −0.6 | 0.941 |
| −0.4 | 0.974 |
| −0.2 | 0.993 |
| 0.2 | 0.993 |
| 0.4 | 0.974 |
| 0.6 | 0.941 |
| 0.8 | 0.897 |

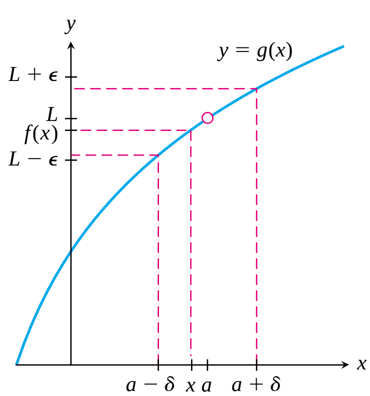


FIGURE 1.57 First we are given $\epsilon > 0$. There is a $\delta > 0$ so that when $0 < |x - a| < \delta$ we have $|g(x) - L| < \epsilon$.

$0 < \theta < 0.5$. Thus we can take $d = 0.5$. For this value of $d$, statement (4) is true. Explain why any positive $d < 0.5$ would also work.

We could also find an acceptable value of $d$ by looking at numerical data. Table 1.10 gives the value of $(\sin\theta)/\theta$ for several $\theta$ values close to 0. The data in Table 1.10 suggest that

$$\left|\frac{\sin\theta}{\theta} - 1\right| < 0.05 \quad \text{when} \quad 0 < |x| < 0.4.$$

(It would be a good idea to verify this graphically. Why?) Based on the data, we can take $d = 0.4$.

The preceding examples illustrate the meaning of "close to" in the definition of *limit*. In this definition, the phrase "we can make $g(x)$ as close to $L$ as we like" means that given any small number $\epsilon$ (pronounced ep′-si-lon), we can make $g(x)$ within $\epsilon$ of $L$, that is, we can make

$$|g(x) - L| < \epsilon. \tag{5}$$

We make $g(x)$ close to $L$ by taking $x$ close to $a$ but not equal to $a$. This means that if $\epsilon$ is given, we can find another number $\delta$ (pronounced del′-ta) so that (5) is true when $0 < |x - a| < \delta$. That is, we must find $\delta$ so that

$$0 < |x - a| < \delta \quad \text{implies} \quad |g(x) - L| < \epsilon.$$

In Example 1b we were given $\epsilon = 0.001$ and found $\delta = d = 0.0002$. In Example 2 we were given $\epsilon = 0.05$ and found $\delta = d = 0.5$. We also showed that $\delta = d = 0.4$ was acceptable.

With this new understanding of "close to," we restate the definition of *limit*.

**DEFINITION** Limit of a Function

Let $a$ and $L$ be real numbers and assume that $g$ is defined throughout an interval centered at $a$, except possibly at the point $x = a$. We say that $g(x)$ has limit $L$ as $x$ approaches $a$ if for each $\epsilon > 0$ there is a corresponding $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \quad \text{then} \quad |g(x) - L| < \epsilon. \tag{6}$$

See Fig. 1.57. We denote this by

$$\lim_{x \to a} g(x) = L.$$

If there is no real number $L$ for which this is true, then we say $g(x)$ has no limit as $x$ approaches $a$, or that $\lim_{x \to a} g(x)$ is undefined.

According to this definition, $a$ may or may not be in the domain of $g$. From (6) we see that we are concerned only with $g(x)$ values for $x$ close to $a$ but not equal to $a$. Hence whether $a$ is in the domain of $g$ or not has nothing to do with the value of $\lim_{x \to a} g(x)$.

## EXAMPLE 3

**a)** Evaluate

$$\lim_{t\to 3} \frac{\sqrt{3t-5}-2}{t-3}. \tag{7}$$

**b)** Let $L$ be the value of the limit in part **a)** and let $\epsilon = 0.035$. Find a number $\delta > 0$ so that

$$\text{if} \quad 0 < |t-3| < \delta, \quad \text{then} \quad \left| \frac{\sqrt{3t-5}-2}{t-3} - L \right| < \epsilon.$$

Use graphical methods.

## Solution

**a)** Notice that when $t = 3$, the numerator and denominator of the fraction

$$\frac{\sqrt{3t-5}-2}{t-3} \tag{8}$$

are both 0. Thus we cannot hope to evaluate (7) by substituting $t = 3$ in (8). To simplify (8), rationalize the numerator by multiplying the numerator and denominator of the expression by $\sqrt{3t-5}+2$:

$$\begin{aligned}
\frac{\sqrt{3t-5}-2}{t-3} &= \frac{\sqrt{3t-5}-2}{t-3} \cdot \frac{\sqrt{3t-5}+2}{\sqrt{3t-5}+2} \\
&= \frac{\left(\sqrt{3t-5}\right)^2 - 2^2}{(t-3)\left(\sqrt{3t-5}+2\right)} \\
&= \frac{3(t-3)}{(t-3)\left(\sqrt{3t-5}+2\right)} \\
&= \frac{3}{\sqrt{3t-5}+2}.
\end{aligned}$$

Thus

$$\lim_{t\to 3} \frac{\sqrt{3t-5}-2}{t-3} = \lim_{t\to 3} \frac{3}{\sqrt{3t-5}+2} = \frac{3}{\sqrt{3\cdot 3-5}+2} = \frac{3}{4}.$$

**b)** With a calculator or CAS, sketch the graph of

$$y = \left| \frac{\sqrt{3t-5}-2}{t-3} - \frac{3}{4} \right|$$

for $t$ values close to 3. See Fig. 1.58. We see for $t \neq 3$ and $2.8 < t < 3.2$, the graph lies below the line $y = \epsilon = 0.035$. Thus we can take $\delta = 0.2$ and see that

$$0 < |t-3| < 0.2 = \delta$$

implies

$$\left| \frac{\sqrt{3t-5}-2}{t-3} - \frac{3}{4} \right| < 0.035 = \epsilon.$$

Note that any positive number smaller than 0.2 would also be an acceptable value for $\delta$.

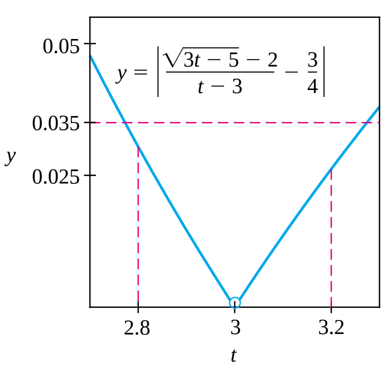


FIGURE 1.58 If $0 < |t-3| < 0.2$, then $\left| \frac{\sqrt{3t-5}-2}{t-3} - \frac{3}{4} \right| < 0.035$.

## Limits and Approximation

When we use a calculator to find the value of sin 1 (where the 1 is in radians), we see the answer 0.841471 on the display. This decimal expression is not the *exact* value of sin 1, but we assume that the answer we see is correct to the number of digits shown. How does the calculator obtain this answer?

In calculators and computers the calculation of the values of square roots, trigonometric functions, logarithms, and exponentials of numbers is based on algorithms and formulas that approximate these functions. To guarantee that the calculator answer is correct to the number of digits shown, the people designing the calculator must know how accurate the approximation is. Limits are important in finding good approximations and in studying the accuracy of an approximation.

The statement

$$\lim_{x\to a} f(x) = L$$

means that $f(x)$ is close to $L$ when $x$ is close to $a$. Suppose that for a given $x$ value we need an approximation of $f(x)$. Would it be acceptable to say

$$f(x) \approx L? \tag{9}$$

We need a lot more information before we can answer this question. For our value of $x$, how good is the approximation given in (9)? How good an approximation do we need? The answer to the second question depends on what we want to do with the approximation. To answer the first question, we need to be able to say something about the *error in the approximation*,

$$E(x) = f(x) - L.$$

We use the error, $E(x)$, to measure how good the approximation (9) is. If $E(x)$ is small enough, the approximation is a good one and may serve our needs. If $E(x)$ is large, the approximation may not be of any use. It is usually impossible or impractical to calculate $E(x)$ exactly, but we can often find a number larger than $|E(x)|$. If this larger number is small, then the error $E(x)$ is even smaller (and could be either positive or negative). In Examples 2 and 3 we learned techniques for finding $\delta$ given $\epsilon$. These same techniques can be used to analyze $E(x)$. In fact, the $E$ in $E(x)$ not only stands for *error*, but also reminds us of $\epsilon$.

**EXAMPLE 4** Because $\lim_{x\to 3} 2^x = 8$, we consider using the approximation

$$2^x \approx 8$$

when $x$ is close to 3. For what values of $x$ is this approximation within 0.01 of the actual value of $2^x$?

### Solution

We need to find $x$ values such that

$$|E(x)| = |2^x - 8| < 0.01.$$

In the language of limits, we are given $\epsilon = 0.01$ and need to find $\delta > 0$ so that

$$\text{if} \quad |x - 3| < \delta, \quad \text{then} \quad |E(x)| = |2^x - 8| < \epsilon = 0.01.$$

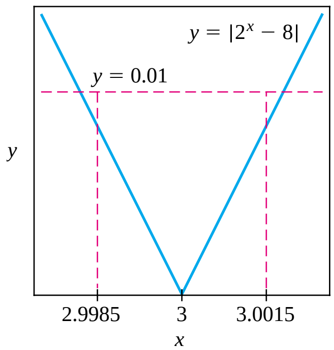


FIGURE 1.59 When $2.9985 < x < 3.0015$ we have $2^x$ within 0.01 of 8.

On your calculator or CAS graph $y = |2^x - 8|$ for $x$ near 3. (Remember, the first graph you produce may not be what you want. The authors used their CAS to plot six different graphs of $y = |2^x - 8|$ before settling on the one in Fig. 1.59.) The graph of $y = |2^x - 8|$ lies below the line $y = 0.01$ for $2.9985 < x < 3.0015$. Hence for $x$ in this interval we can say

$$2^x \approx 8$$

with an error of at most 0.01.

Sometimes we can manipulate a limit result to get a useful approximation. For example, in the preceding section we showed that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We use this result to find a good approximation to $\sin x$ for small $x$. Let

$$E(x) = \frac{\sin x}{x} - 1. \qquad (10)$$

Because

$$\lim_{x\to 0} E(x) = \lim_{x\to 0}\left(\frac{\sin x}{x} - 1\right) = 1 - 1 = 0,$$

we know that $E(x)$ can be made small by taking $x$ close to 0. Rewrite (10) as

$$\frac{\sin x}{x} = 1 + E(x).$$

Then multiply by $x$ to get

$$\sin x = x + x \cdot E(x),$$

or

$$\sin x - x = x \cdot E(x). \qquad (11)$$

Hence

$$\sin x \approx x$$

because the error $\sin x - x = x \cdot E(x)$ is small when $x$ is small. We say more about this error in the next example.

**EXAMPLE 5** Discuss the error in the approximation

$$\sin x \approx x \qquad (12)$$

for $-0.5 < x < 0.5$. (Make sure your calculator is in radian mode.)

### Solution

In Example 2 we showed that

$$|E(x)| = \left|\frac{\sin x}{x} - 1\right| < 0.05$$

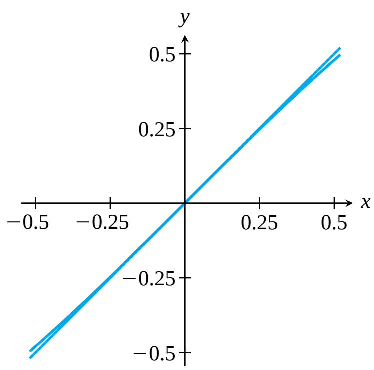


FIGURE 1.60 The graphs of $y = \sin x$ and $y = x$ are close for $-0.5 < x < 0.5$.

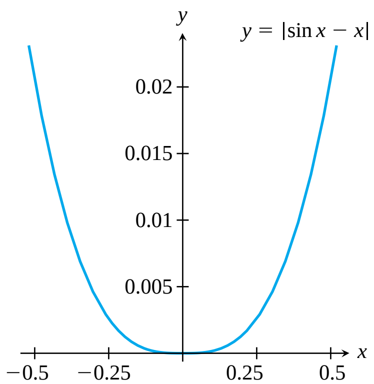


FIGURE 1.61 The graph of $y = |E(x)| = |\sin x - x|$ shows the error is small for $-0.5 < x < 0.5$.

if $-0.5 < x < 0.5$. Using (11), we see that for $x$ in this interval, the error in the approximation $\sin x \approx x$ is

$$|\sin x - x| = |x \cdot E(x)| = |x| \cdot |E(x)| \le 0.05|x| \le 0.05 \cdot 0.5 = 0.025. \quad (13)$$

Equation (13) tells us that if we use the approximation $\sin x \approx x$ for $-0.5 < x < 0.5$, then the error will be no more than 0.025 in magnitude. From (13) we also see that for $-0.5 < x < 0.5$,

$$|\sin x - x| \le 0.05|x|.$$

This means that for such $x$, the error in the approximation (12) is never more than 5 percent of $|x|$. Thus the error gets smaller as $|x|$ gets smaller, and the error is small compared to $|x|$.

To further understand the error, we graph $y = \sin x$ and $y = x$ for $-0.5 < x < 0.5$ on the same set of coordinate axes. The graph is shown in Fig. 1.60. The two graphs are close, which means that the numbers $\sin x$ and $x$ are close for $-0.5 < x < 0.5$.

For another look at the error, graph the equation $y = |E(x)| = |\sin x - x|$ on $-0.5 < x < 0.5$. The graph is shown in Fig. 1.61. This graph shows plainly that the error in the approximation gets small as $|x|$ gets small.

For a more specific illustration, we compare $\sin x$ and $x$ for a small value of $x$, say $x = 0.3$. A calculator gives $\sin(0.3) \approx 0.295520$, a value close to $x = 0.3$.

**EXAMPLE 6** Evaluate

$$\lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1}$$

and use the result to find a good approximation to $\sqrt{x}$ for $x$ close to 1. Discuss the error in the approximation for $x$ in the interval $0.9 \le x \le 1.1$.

## Solution

For $x \ne 1$ we have

$$\frac{\sqrt{x} - 1}{x - 1} = \frac{\sqrt{x} - 1}{(\sqrt{x})^2 - 1^2} = \frac{\sqrt{x} - 1}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \frac{1}{\sqrt{x} + 1}.$$

Therefore

$$\lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x\to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2}.$$

Now let

$$\frac{\sqrt{x} - 1}{x - 1} - \frac{1}{2} = E(x). \quad (14)$$

Solve this equation for $\sqrt{x}$. To do this, add $\frac{1}{2}$ to both sides of (14), then multiply both sides of the result by $x - 1$, and then add 1 to both sides. We have

$$\sqrt{x} = 1 + \frac{1}{2}(x - 1) + E(x)(x - 1). \quad (15)$$

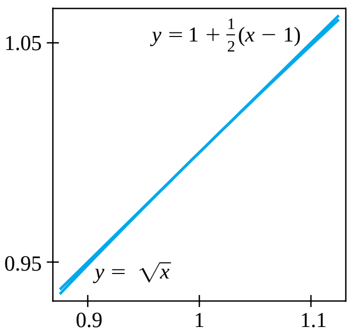


FIGURE 1.62 The graphs of $y = \sqrt{x}$ and $y = 1 + \frac{1}{2}(x - 1)$ are so close we see little difference.

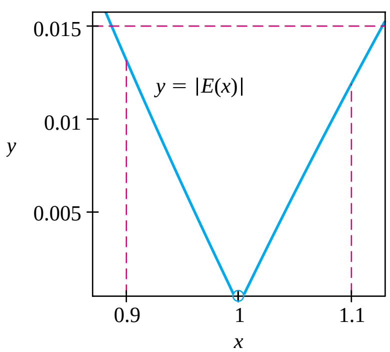


FIGURE 1.63 The graph shows that $y = |E(x)| = \left|\frac{\sqrt{x}-1}{x-1} - \frac{1}{2}\right| < 0.015$ when $0.9 < x < 1.1$.

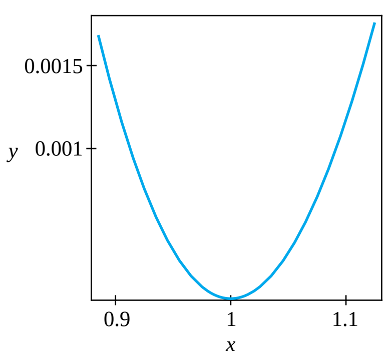


FIGURE 1.64 The graph of $y = \left|\sqrt{x} - \left(1 + \frac{1}{2}(x - 1)\right)\right|$ shows that the error is very small for $0.9 < x < 1.1$.

When $x$ is close to 1, we know that $E(x)$ is close to 0 and that $(x - 1)$ is small. Hence for $x$ close to 1, $E(x)\,(x - 1)$ is small. If we drop this small term from (15), we get an approximation for $\sqrt{x}$:

$$\sqrt{x} \approx 1 + \frac{1}{2}(x - 1). \tag{16}$$

Figure 1.62 shows the graphs of $y = \sqrt{x}$ and $y = 1 + \frac{1}{2}(x - 1)$ on the interval $[0.9, 1.1]$. Apparently the approximation (16) is quite good.

From (15), the absolute value of the error in this approximation is

$$\left|\sqrt{x} - \left(1 + \frac{1}{2}(x - 1)\right)\right| = |(x - 1)E(x)|. \tag{17}$$

To estimate this error for $0.9 \le x \le 1.1$, recall (14) and graph

$$y = |E(x)| = \left|\frac{\sqrt{x} - 1}{x - 1} - \frac{1}{2}\right|$$

on this interval. The graph appears in Fig. 1.63 and shows that $|E(x)| < 0.015$ when $0.9 \le x \le 1.1$. Use this overestimate for $|E(x)|$ in (17) to see that

$$\left|\sqrt{x} - \left(1 + \frac{1}{2}(x - 1)\right)\right| \le 0.015|x - 1|.$$

Hence the error in the approximation (16) is at most 1.5% of the value of $|x - 1|$. Furthermore, for $0.9 \le x \le 1.1$, we have

$$|(x - 1)E(x)| \le (0.1)\,(0.015) = 0.0015,$$

so the error in (16) is no more than 0.0015. In Fig. 1.64 we show the graph of $y = |(x - 1)E(x)|$. This graph shows that on the interval $0.9 \le x \le 1.1$ the error in the approximation is less than 0.0015. For a more specific illustration, we use (16) to approximate $\sqrt{0.95}$. Putting $x = 0.95$ into (16),

$$\sqrt{0.95} \approx 1 + \frac{1}{2}(0.95 - 1) = 0.975,$$

while a calculator gives a value of 0.974679. Not bad! In Exercise 40 we do more work with this example to obtain a better approximation to the square root function.

## One-Sided Limits

According to Einstein's special theory of relativity, the length of a rod measured at rest is different from the length measured when the rod is in motion. If you measure an arrow on your desk and find it has length $L_0$, and then measure the same arrow as it flies by at speed $v$, the length will be

$$L = L_0\sqrt{1 - \left(\frac{v}{c}\right)^2}, \tag{18}$$

where $c = 2.998 \times 10^{10}$ cm/s is the speed of light. At the everyday speeds $v$ of cars and planes, this effect is not noticeable, but if $v$ is close to $c$, the difference between $L_0$ and $L$ can be significant.

Although an arrow can never travel at the speed of light, physicists are interested in studying objects at speeds close to $c$. This suggests taking the limit of (18) as $v$ approaches $c$. However, because (18) is defined for $0 < v < c$ but not for $v > c$, we really want to study (18) only for speeds $v$ close to but less than $c$. In this case, we say that we are evaluating the limit of (18) as $v$ approaches $c$ from the left.

**DEFINITION** Left-hand Limit

Let $g(x)$ be defined in an interval $a - r < x < a$ for some $r > 0$. We say that $g(x)$ has **left-hand limit** $L$ as $x$ approaches $a$ if we can make $g(x)$ as close to $L$ as we like by taking $x < a$ and close to $a$. We write

$$\lim_{x \to a^-} g(x) = L.$$

This is read "the limit of $g(x)$ as $x$ approaches $a$ from the left is $L$."

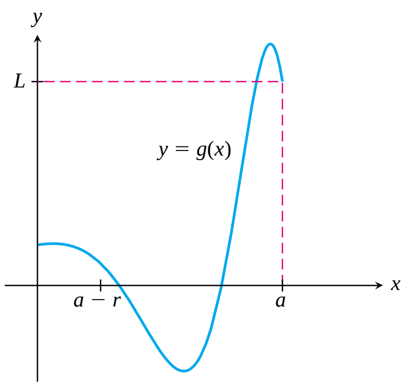


FIGURE 1.65 When $x < a$ and $x$ is close to $a$, $g(x)$ is close to $L$. We write $\lim_{x \to a^-} g(x) = L$.

We illustrate the idea of the left-hand limit in Fig. 1.65. The right-hand limit, $\lim_{x \to b^+} g(x)$, is defined similarly. See Exercise 26.

**EXAMPLE 7** Find the limit of the length of the flying arrow as $v$ approaches $c$ from the left.

**Solution**

When $v < c$ and close to $c$, expression (18) is defined. For such values of $v$,

$$L = L_0\sqrt{1 - \left(\frac{v}{c}\right)^2} \quad \text{is close to} \quad L_0\sqrt{1 - \left(\frac{c}{c}\right)^2} = 0.$$

Hence

$$\lim_{v \to c^-} L_0\sqrt{1 - \left(\frac{v}{c}\right)^2} = 0.$$

This means that as the speed of the arrow approaches the speed of light, the length of the arrow approaches 0.

**EXAMPLE 8** The function $f$ is defined by

$$f(x) = \begin{cases} -2x + 1 & x < 1 \\ 3 & x = 1 \\ 2x^2 & x > 1. \end{cases}$$

Evaluate

$$\lim_{x \to 1^-} f(x), \quad \lim_{x \to 1^+} f(x), \quad \text{and} \quad \lim_{x \to 1} f(x).$$

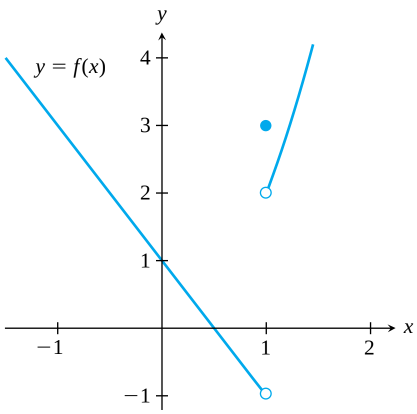


FIGURE 1.66 The graph of $y = f(x)$.

**Solution**

The graph of $y = f(x)$ is shown in Fig. 1.66. From the graph we see that if $x$ is close to 1 and less than 1, then $f(x)$ is close to $-1$. In fact, when $x < 1$, we have $f(x) = -2x + 1$, so

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(-2x + 1) = -2 \cdot 1 + 1 = -1.$$

When $x > 1$, we have $f(x) = 2x^2$. Hence

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x^2 = 2 \cdot 1^2 = 2.$$

This can also be seen by looking at the graph for $x > 1$ and close to 1. This discussion also shows that when $x$ is close to 1, then $f(x)$ might be close to $-1$ or close to 2, depending on whether $x < 1$ or $x > 1$. Because the values of $f(x)$ do not get close to one and only one number $L$ as $x$ approaches 1, we conclude that although the one-sided limits exist at $x = 1$,

$$\lim_{x\to 1} f(x) \quad \text{does not exist.}$$

## Continuous Functions

In Section 1.5 we saw that if $p(x)$ is a polynomial and $c$ is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

In fact, we have seen many examples with

$$\lim_{x\to c} f(x) = f(c).$$

Functions with such "well-behaved" limits are very important.

**DEFINITION** Continuous Function

The function $f$ is **continuous at** $c$ if $f$ is defined at $c$ and

$$\lim_{x\to c} f(x) = f(c).$$

See Fig. 1.67. If $f$ is continuous at every point in an interval $a < x < b$, we say that $f$ is **continuous on the interval** $a < x < b$. If $f$ is continuous at every point in its domain, then we say that $f$ is **continuous.**

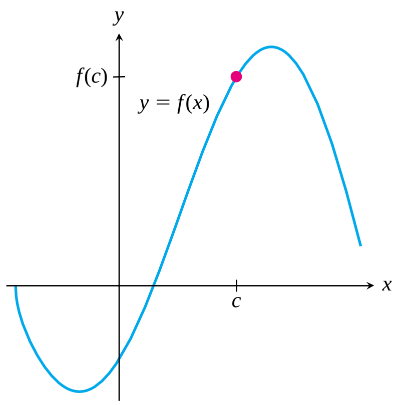


FIGURE 1.67 If $\lim_{x\to c} f(x) = f(c)$, then $f$ is continuous at $c$.

When we look closely at this definition, we see that three things must be true for a function $f$ to be continuous at $c$:

**a)** $f(x)$ must be defined at $x = c$.
**b)** $\lim_{x\to c} f(x)$ must exist.
**c)** The value of the limit in b is $f(c)$.

In Example 1 of Section 1.5 we showed that

$$\lim_{x\to 5}(2x^2 + 4) = 2 \cdot 5^2 + 4.$$

Hence the function defined by $2x^2 + 4$ is continuous at $x = 5$. In Example 2 of Section 1.5 we showed that

$$\lim_{\theta \to \pi/4} \cos \theta = \cos(\pi/4).$$

This means that the cosine function is continuous at $\theta = \pi/4$.

If one or more of the conditions a, b, c is not satisfied, then $f(x)$ is **discontinuous** (or not continuous) at $x = c$. See Figs. 1.68 and 1.69. We have already seen many examples of discontinuity. For example, we have seen that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

However, the function defined by $(\sin x)/x$ is not continuous at $x = 0$ because the function is not defined for this value of $x$.

The function $f$ of Example 8 is defined at $x = 1$ but is not continuous at this point because

$$\lim_{x \to 1} f(x) \quad \text{does not exist.}$$

As mentioned earlier, if $p$ is any polynomial function

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

then for any real number $c$,

$$\lim_{x \to c} p(x) = p(c).$$

This means that a polynomial is continuous at every point in its domain. Hence any polynomial is a continuous function.

In Section 1.5 we also saw that for any real number $c$

$$\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c.$$

Thus the sine function and the cosine function are continuous.

In addition, the rules for combining limits given in the previous section imply that most combinations of continuous functions are continuous. Likewise, the result on limits and compositions shows that compositions of continuous functions are also continuous.

Continuous functions have many important and interesting properties. In one sense, the graph of a continuous function is well behaved.

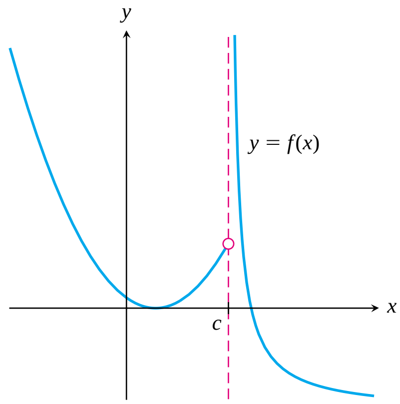


FIGURE 1.68 If $f$ is not defined at $c$ or $\lim_{x \to c} f(x)$ does not exist, then $f$ is discontinuous at $c$.

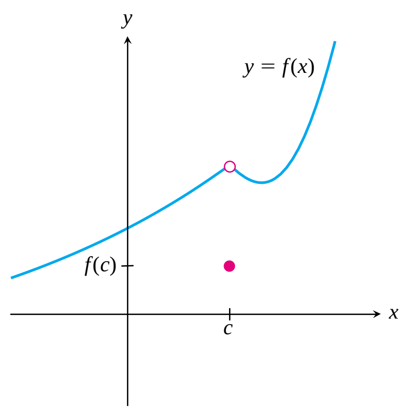


FIGURE 1.69 If $\lim_{x \to c} f(x)$ exists but is not equal to $f(c)$, then $f$ is discontinuous at $c$.

**Graphs of Continuous Functions**

Let $f$ be continuous on an interval $a < x < b$. Then the graph of $y = f(x)$ on $a < x < b$ is an unbroken curve.

According to this statement, the graph of a function continuous on an interval cannot have jumps or breaks like those seen in Fig. 1.68, Fig. 1.69, or Fig. 1.50. The converse of this result is "almost" true. A precise statement of the converse would take some terminology not usually encountered in calculus courses. However, for our purposes, the following statement is enough.

**Graphs Representing Continuous Functions**

For a function $f$, suppose that the graph $y = f(x)$ for $a < x < b$ is an unbroken curve and the graph exhibits no oscillatory behavior like that seen in the third graph in Fig. 1.50. Then $f$ is continuous on the interval $a < x < b$.

Thus the graph in Fig. 1.67 is the graph of a continuous function.

## Exercises 1.6

**1.** Let $f(x) = 3x^2 - 2x + 1$.

T **a.** Find the value of $\lim_{x \to -1} f(x)$.

**b.** Let $L$ be the value of the limit found in part a and let $\epsilon = 0.1$. Use graphical methods to find a $\delta > 0$ so that if $0 < |x - (-1)| < \delta$, then $|f(x) - L| < \epsilon$.

**2.** Let $f(x) = x + 2\cos x$.

T **a.** Find the value of $\lim_{x \to \pi} f(x)$.

**b.** Let $L$ be the value of the limit found in part a and let $\epsilon = 0.01$. Use graphical methods to find a $\delta > 0$ so that if $0 < |x - \pi| < \delta$, then $|f(x) - L| < \epsilon$.

**3.** Let $r(\theta) = 6$.

**a.** Find the value of $\lim_{\theta \to \sqrt{5}} r(\theta)$.

**b.** Let $L$ be the value of the limit found in part a and let $\epsilon = 0.1$. Use graphical methods to find a $\delta > 0$ so that if $0 < \left|\theta - \sqrt{5}\right| < \delta$, then $|r(\theta) - L| < \epsilon$.

**4.** Let $g(t) = 2^t$.

T **a.** Find the value of $\lim_{t \to 1} g(t)$.

**b.** Let $L$ be the value of the limit found in part a and let $\epsilon = 0.05$. Use graphical methods to find a $\delta > 0$ so that if $0 < |t - 1| < \delta$, then $|g(t) - L| < \epsilon$.

**5.** Let $H(r) = \dfrac{r^3 + 8}{r^2 - 4}$.

T **a.** Find the value of $\lim_{r \to -2} H(r)$.

**b.** Let $L$ be the value of the limit found in part a and let $\epsilon = 0.1$. Use graphical methods to find a $\delta > 0$ so that if $0 < |r - (-2)| < \delta$, then $|H(r) - L| < \epsilon$.

**6.** Let $t(\theta) = \dfrac{\cos\theta - 1}{2\theta^2}$.

T **a.** Find the value of $\lim_{\theta \to 0} t(\theta)$.

**b.** Let $L$ be the value of the limit found in part a and let $\epsilon = 0.005$. Use graphical methods to find a $\delta > 0$ so that if $0 < |\theta - 0| < \delta$, then $|t(\theta) - L| < \epsilon$.

*Exercises 7–13: A function f and a number a are given.*

**a.** Find the value of $\lim_{x \to a} f(x)$. Use analytical, numerical, or graphical means.

**b.** Let $L$ be the value of the limit found in part a. Find an interval containing the point $x = a$ on which $f(x) \approx L$ with an error of at most 0.01.

T **7.** $f(x) = -4x^2 + 2x - 1, \quad a = 2$

T **8.** $f(x) = \dfrac{1}{\sqrt{x + 1}}, \quad a = 1$

T **9.** $f(x) = \cos(2x), \quad a = 0$

T **10.** $f(x) = \dfrac{10^x - 1}{x}, \quad a = 0$

T **11.** $f(x) = \dfrac{x - 5}{\sqrt{x} - \sqrt{5}}, \quad a = 5$

T **12.** $f(x) = \dfrac{\tan x - 1}{x - \pi/4}, \quad a = \pi/4$

T **13.** $f(x) = \dfrac{1/(x + 1) - \frac{1}{2}}{x - 1}, \quad a = 1$

T **14. a.** Evaluate

$$\lim_{x \to 2} \frac{3x^2 - 2x - 8}{x - 2}.$$

**b.** Use the result of part **a** to find an approximation to $3x^2 - 2x - 8$ for $x$ values near 2.

**c.** Find an interval containing 2 on which the error in this approximation is less than 0.001.

**d.** For $x = 1.98$, compare the actual value of $3x^2 - 2x - 8$ with the value given by the approximation.

T **15. a.** Evaluate (or estimate)

$$\lim_{r\to-2}\frac{\sqrt{2r^2+1}-3}{r+2}.$$

b. Use the result of part **a** to find an approximation to $\sqrt{2r^2+1}$ for $r$ values near $-2$.

c. Find an interval containing $-2$ on which the error in this approximation is less than 0.001.

T d. For $r=-1.98$, compare the actual (calculator) value of $\sqrt{2r^2+1}$ with the value given by the approximation.

**16.** a. Use graphical or numerical means to find the value of

$$\lim_{t\to2}\frac{3^t-3^2}{t-2}.$$

b. Use the result of part **a** to find an approximation to $3^t$ for $t$ values near 2.

c. Find an interval containing 2 on which the error in this approximation is less than 0.01.

T d. For $t=2.015$, compare the actual (calculator) value of $3^t$ with the value given by the approximation.

**17.** a. Use graphical or numerical means to find the value of

$$\lim_{x\to\pi/4}\frac{\tan x-1}{x-\pi/4}.$$

b. Use the result of part **a** to find an approximation to $\tan x$ for $x$ values near $\pi/4$.

c. Find an interval containing $\pi/4$ on which the error in this approximation is less than 0.001.

d. For $x=0.75$, compare the actual value of $\tan x$ with the value given by the approximation.

**18.** In Fig. 1.60, which is the graph of $y=x$ and which is the graph of $y=\sin x$? Give reasons for your answer.

**19.** Let $h(t)=t/|t|$.

a. Evaluate $\lim_{t\to0^+}h(t)$ and $\lim_{t\to0^-}h(t)$.

b. What can be said about $\lim_{t\to0}h(t)$?

c. Is $h(t)$ continuous at $t=0$? Why or why not?

**20.** Let $P(w)$ be the cost of first-class postage for a letter that weighs $w$ ounces. If $0<w<1$, then $P(w)=34¢$. If $w>1$, the cost is 34¢ plus 22¢ for each ounce or fraction of an ounce above 1.

a. Sketch a graph of the function $C=P(w)$ for $0<w<5.5$.

b. Evaluate $\lim_{w\to2^-}P(w)$ and $\lim_{w\to2^+}P(w)$.

c. Tell why $P(w)$ is discontinuous for $w=1,2,3,4,\ldots$.

**21.** The graph of a function $f$ is shown in the accompanying figure.

a. Evaluate $\lim_{x\to-1^+}f(x)$ and $\lim_{x\to-1^-}f(x)$.

b. Evaluate $\lim_{x\to0^+}f(x)$ and $\lim_{x\to0^-}f(x)$.

c. Evaluate $\lim_{x\to1^+}f(x)$ and $\lim_{x\to1^-}f(x)$.

d. For what values of $x$ between $-3$ and 3 is $f$ discontinuous?

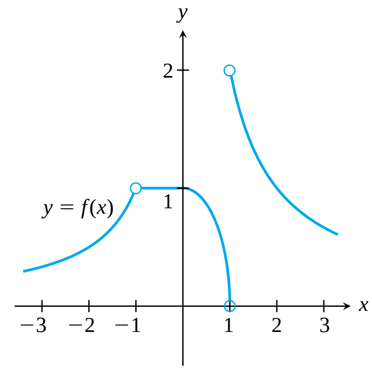


Figure for Exercise 21.

**22.** The graph of a function $g$ is shown in the accompanying figure.

a. Evaluate $\lim_{x\to-4^+}g(x)$ and $\lim_{x\to-4^-}g(x)$.

b. Evaluate $\lim_{x\to-2^+}g(x)$ and $\lim_{x\to-2^-}g(x)$.

c. Evaluate $\lim_{x\to2^+}g(x)$ and $\lim_{x\to2^-}g(x)$.

d. For what values of $x$ between $-5$ and 3 is $g$ discontinuous?

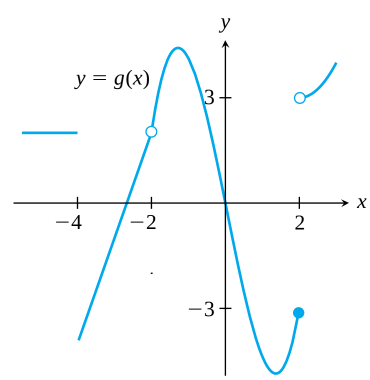


Figure for Exercise 22.

**23.** Restate the rule for limits and compositions in Section 1.5 in such a way that the conditions on the function $h$ are expressed in terms of the continuity of $h$ at $L$.

**24.** a. For $x\neq0$ let

$$g(x)=x\sin\left(\frac{1}{x}\right).$$

Show that $\lim_{x\to0}g(x)=0$.

b. For $x\neq0$ let

$$h(x) = \frac{\sin x}{x}.$$

Note that $h(x)$ is not defined for $x = 0$ but that $\lim_{x\to 0} h(x) = 1$. Show, however, that $\lim_{x\to 0} h(g(x))$ does not exist.

c. Carefully tell why this example shows that it is necessary to have $h$ defined at $L$ in the rule for limits and compositions on page 43.

**25.** Give an $\epsilon$, $\delta$ definition for the left-hand limit. Model it on the $\epsilon$, $\delta$ definition for the limit given in this section.

**26.** Write a definition for the right-hand limit of a function. Model your definition on the definition of the left-hand limit given in this section.

**27.** In this exercise we "verify" equation (13) from the rules for combining limits in Section 1.5. Let $g$ and $h$ be functions and suppose that

$$\lim_{x\to a} g(x) = 3, \quad \lim_{x\to a} h(x) = 5.$$

Take $\epsilon = 0.1$. Use the following two steps as a guide to argue why there must be a $\delta > 0$ so that $0 < |x - a| < \delta$ implies

$$|(g(x) + h(x)) - (3 + 5)| < \epsilon.$$

a. First tell why there must be a $\delta > 0$ so that when $0 < |x - a| < \delta$ we have

$$|g(x) - 3| < 0.05$$

and

$$|h(x) - 5| < 0.05.$$

b. Let $\delta$ have the value discussed in part a. Show that $0 < |x - a| < \delta$ implies

$$|(g(x) + h(x)) - 8| < 0.1 = \epsilon.$$

**28.** In this exercise we "verify" equation (15) from the rules for combining limits in Section 1.5. Let $g$ and $h$ be functions and suppose that

$$\lim_{x\to a} g(x) = 3, \quad \lim_{x\to a} h(x) = 5.$$

Take $\epsilon = 0.1$. Use the following two steps as a guide to argue why there must be a $\delta > 0$ so that $0 < |x - a| < \delta$ implies

$$\left|\frac{g(x)}{h(x)} - \frac{3}{5}\right| < \epsilon.$$

a. First tell why there must be a $\delta > 0$ so that when $0 < |x - a| < \delta$ we have

$$|g(x) - 3| < 0.25 \quad \text{and} \quad |h(x) - 5| < 0.25.$$

b. Let $\delta$ have the value discussed in part a. Show that $0 < |x - a| < \delta$ implies

$$\left|\frac{g(x)}{h(x)} - \frac{3}{5}\right| < 0.1 = \epsilon.$$

**29.** Let

$$p(x) = x^3 - 4x + 6.$$

The following steps outline a method for using the squeeze theorem (see Exercise 45 of Section 1.5) to show that $\lim_{x\to 1} p(x) = p(1) = 3$.

a. Note that

$$p(x) - p(1)$$

takes the value 0 when $x = 1$. Hence the polynomial $p(x) - p(1)$ has a factor of $x - 1$. Verify that

$$p(x) - p(1) = (x - 1)(x^2 + x - 3).$$

b. For $|x - 1| < 1$ (i.e., $0 < x < 2$) show that

$$|x^2 + x - 3| < 9.$$

c. Show that for $|x - 1| < 1$,

$$|p(x) - 3| \le 9|x - 1|.$$

d. Use the squeeze theorem and part **c** to show that

$$\lim_{x\to 1} p(x) = p(1) = 3.$$

Relate this result to the rule for limits of polynomials, sines, and cosines on page 42 in Section 1.5.

**30.** Use the technique outlined in the previous problem to show that

$$\lim_{x\to 2}(-2x^3 + 3x^2 + x - 3) = -5.$$

**31.** Let $p(x)$ be a polynomial and $a$ a real number. Based on the outline given in Exercise 29, discuss a method for showing that $\lim_{x\to a} p(x) = p(a)$. Note that this gives a method of proving equation (16) of the rule for limits of polynomials, sines, and cosines on page 42 in Section 1.5.

**32.** Define $r = r(c)$ to be the number of distinct real zeros of the polynomial

$$x^2 - 3x + c.$$

For example, $r(-4) = 2$ because $x^2 - 3x - 4 = 0$ has two different solutions, $x = 4$ and $x = -1$. However, $r(5) = 0$ because the equation $x^2 - 3x + 5 = 0$ has no real solutions. For what values of $c$ is $r(c) = 2$? When is $r(c) = 1$? When is $r(c) = 0$? Graph the function $r = r(c)$ and list the values of $c$ where the function $r$ is not continuous.

**33.** Define $r = r(b)$ to be the number of distinct real zeros of the polynomial

$$x^2 + bx + 4.$$

For example, $r(3) = 0$ because $x^2 + 3x + 4 = 0$ has no real solutions, and $r(5) = 2$ because the equation $x^2 + 5x + 4 = 0$ has two real solutions. For what values of $b$ is $r(b) = 2$? When is $r(b) = 1$? When is $r(b) = 0$? Graph the function $r = r(b)$ and list the values of $b$ where the function $r$ is not continuous.

**34.** Define $r(d)$ to be the number of distinct real zeros of the polynomial

$$2x^3 + 3x^2 - 12x + d.$$

For example, $r(0) = 3$ because $2x^3 + 3x^2 - 12x + 0 = 0$ has three real solutions, as shown in the accompanying figure. For what values of $d$ is $r(d) = 3$? When is $r(d) = 2$? When is $r(d) = 1$? When is $r(d) = 0$? Graph the function $r = r(d)$ and list the values of $d$ for which $r$ is not continuous.

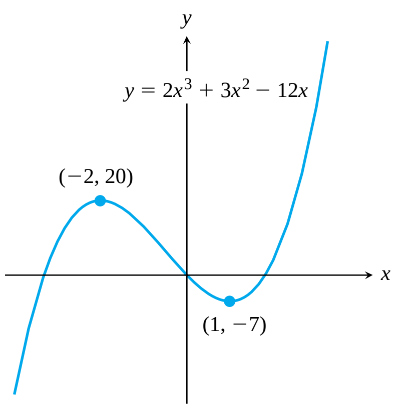


Figure for Exercise 34.

**35.** Find the value of

$$\lim_{x\to 8} \frac{\sqrt[3]{x} - 2}{x - 8}.$$

Use this result to develop an approximation to $\sqrt[3]{x}$ for $x$ values close to 8. Find an interval containing 8 on which the error in the approximation is less than 1 percent of the value of $|x - 8|$.

**36.** Use graphical or numerical means to find the value of

$$\lim_{x\to 3} \frac{2^x - 8}{x - 3}.$$

Use this result to develop an approximation to $2^x$ for $x$ values close to 3. Find an interval containing 3 on which the error in the approximation is less than 1 percent of the value of $|x - 3|$.

**37.** Find the value of

$$\lim_{x\to -2} \frac{\dfrac{1}{\sqrt{x^2+5}} - \dfrac{1}{3}}{x + 2}.$$

Use this result to develop an approximation to $1/\sqrt{x^2+5}$ for $x$ values close to $-2$. Find an interval containing $-2$ on which the error in the approximation is less than 1 percent of the value of $|x - (-2)|$.

**38.** Use graphical or numerical means to find the value of

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta^2}$$

(or see Exercise 14 in Section 1.5). Use this result to find an approximation to $\cos\theta$ for $\theta$ values close to 0. Find an interval containing 0 on which the error in the approximation is less than 1 percent of the value of $\theta^2$. On this interval, compare $\cos\theta$ and the approximation graphically.

**39.** Use graphical or numerical means to find the value of

$$\lim_{\theta\to 0} \frac{\sin\theta - \theta}{\theta^3}.$$

Use this result to find an approximation to $\sin\theta$ for $\theta$ values close to 0. Find an interval containing 0 on which the error in the approximation is less then 1 percent of the value of $\theta^3$. Compare $\sin\theta$ and the approximation graphically.

**40.** In Example 6 we showed that

$$\sqrt{x} \approx 1 + \frac{1}{2}(x - 1).$$

We use this result to derive a better approximation to the square root function.

**a.** Verify that

$$\lim_{x\to 1} \frac{\sqrt{x} - \left(1 + \frac{1}{2}(x-1)\right)}{(x-1)^2} = -\frac{1}{8}.$$

**b.** Use the result of part **a** to derive an approximation for $\sqrt{x}$.

**c.** Discuss the error in this approximation on the interval $0.9 \le t \le 1.1$ and compare the error to the approximation found in Example 6.

**41.** Suppose that $\lim_{x\to c} f(x) = L$. What can be said about

$$\lim_{x\to c^-} f(x) \quad \text{and} \quad \lim_{x\to c^+} f(x)?$$

**42.** Suppose that

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = M.$$