1.5 Limits

The 1997 Nobel Prize in physics was awarded to Steven Chu of Stanford University, William D. Phillips of the National Institute of Standards and Technology, and Claude Cohen-Tannoudji of the Collège de France and École Normale Supérieure in recognition of their development of a technique for slowing and trapping atoms.

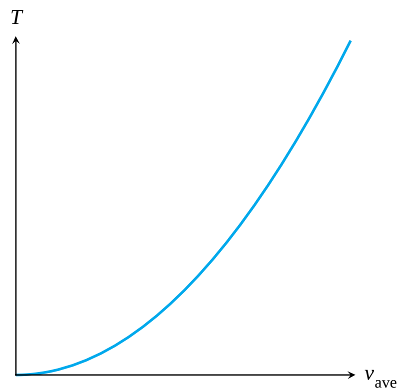

FIGURE 1.40 Temperature T, measured in Kelvins, is a function of average molecular speed, v_{ave}.

At room temperature, atoms and molecules in the air move with speeds of up to 1000 meters per second. Using the methods developed by the three Nobel laureates, scientists have been able to use lasers to slow individual atoms to speeds of less than 1 meter per second and to keep the slowed atoms confined using blasts of laser light and magnetic traps. When the atoms of a gas move at such low speeds, the temperature of the gas is very close to absolute zero (0 Kelvin). See Fig. 1.40. In fact, using the techniques described, temperatures of just a few millionths of a degree above absolute zero have been attained.

Scientists know that a temperature of absolute zero can never be achieved, but by studying slowed atoms they can learn more about the properties of matter at low temperatures, inferring what might happen if even lower temperatures were achieved. In mathematical language, we might say that physicists are investigating "the limit of properties of matter as temperature approaches absolute zero."

When we compute the rate of change of a function f at a point $x = a$ in its domain, we are in a situation similar to that of the physicist studying properties of matter at low temperatures. The rate of change is found by investigating the behavior of the quotient

$$\frac{f(a + h) - f(a)}{h} \tag{1}$$

when h is close to 0. We cannot simply set $h = 0$ in (1) because the expression is undefined for such h. Instead, we investigate the value of (1) for real numbers h close to 0. We might do so numerically (with a table of values), graphically, algebraically, or by combining two or more of these approaches. In doing this, our goal is to obtain a number or expression that we call the rate of change of f with respect to x at $x = a$. This number or expression is *the limit of* (1) *as h approaches 0.*

The definition of *limit* is motivated by the need to study the behavior of expressions like (1) near a value of h where the expression may not be defined.

DEFINITION Limit of a Function

Let g be a function and let a and L be real numbers. If we can make $g(x)$ as close to L as we like by taking x close to a, but not equal to a, then we say

$g(x)$ has **limit** L as x approaches a.

We denote this by

$$\lim_{x \to a} g(x) = L. \tag{2}$$

Equation (2) is read "the limit of $g(x)$ as x approaches a equals L."

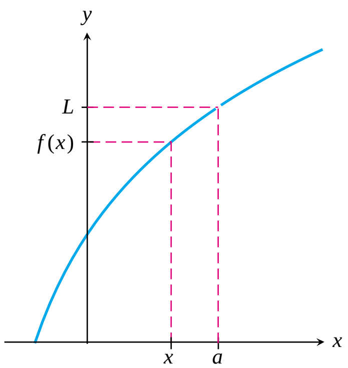

FIGURE 1.41 When x is close to a, $f(x)$ is close to L. We write $\lim_{x\to a} f(x) = L$.

This definition of *limit* is meant to be a "working" definition. By thinking of $L = \lim_{x\to a} g(x)$ as the number that $g(x)$ is close to when x is close to a, as illustrated in Fig. 1.41, we can evaluate most limits, including limits of expressions like (1). However, we will need to do more with limits than simply evaluate limits of functions. We will use limits to study rates of change, approximations, areas, and many other things. For this we will need a more precise definition than the one given. In the next section, we will see that mathematicians mean something very special when they say "close to." Once "close to" is properly defined, we will have a definition of *limit* that is good for more than just evaluation of limits.

Many limits can be evaluated using little more than common sense and knowledge about some familiar functions.

EXAMPLE 1 Evaluate $\lim_{x\to 5}(2x^2 + 4)$.

Solution

We know that when x is close to 5, x^2 is close to $5^2 = 25$. It follows that when x is close to 5, $2x^2 + 4$ is close to $2\cdot 5^2 + 4 = 54$. Hence

$$\lim_{x\to 5}(2x^2 + 4) = 2\cdot 5^2 + 5 = 54. \tag{3}$$

The graph of $y = 2x^2 + 4$ near $x = 5$ is shown in Fig. 1.42. From the graph we see that when x is close to 5, the value of $y = 2x^2 + 4$ is close to 54. This is consistent with (3).

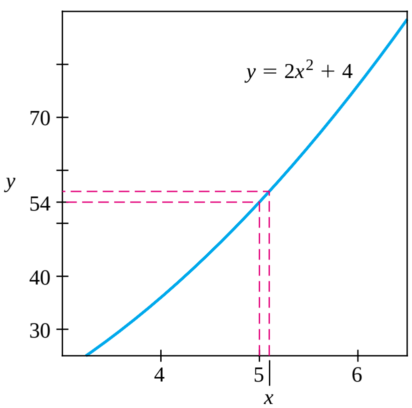

FIGURE 1.42 When x is close to 5, $2x^2 + 4$ is close to 54.

EXAMPLE 2 Find the value of $\lim_{\theta\to\pi/4} \cos\theta$.

Solution

When θ is close to $\pi/4$, $\cos\theta$ is close to $\cos\pi/4 = \sqrt{2}/2$. Therefore

$$\lim_{\theta\to\pi/4} \cos\theta = \cos(\pi/4) = \frac{\sqrt{2}}{2}.$$

This is illustrated by looking at the graph of $y = \cos\theta$ near $\theta = \pi/4$. In Fig. 1.43 we see that if θ is close to $\pi/4$, then $\cos\theta$ is close to $\sqrt{2}/2$.

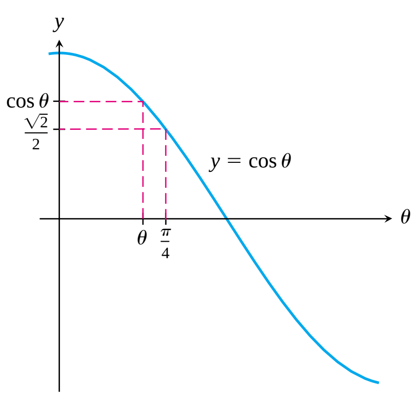

FIGURE 1.43 When θ is close to $\pi/4$, $\cos\theta$ is close to $\sqrt{2}/2$.

More care is needed in evaluating a limit near a point where the function is not defined. In these cases the limit cannot be determined by a simple substitution as in the previous examples.

EXAMPLE 3 Evaluate

$$\lim_{t\to 3}\frac{2t^2 - 4t - 6}{t^2 - 9}. \tag{4}$$

Solution

We demonstrate three techniques for exploring this limit: numerical, graphical, and analytical. First, however, let's see why the "commonsense" approach used in the two previous examples fails here. If t is close to but not equal to 3, then the numerator in (4) is close to 0 because

$$2t^2 - 4t - 6 \approx 2(3)^2 - 4(3) - 6 = 0.$$

Similarly, if t is close to 3, then the denominator in (4) is also close to 0 because

$$t^2 - 9 \approx 3^2 - 9 = 0.$$

Java Applet
Limits
Explores the limit of a function $f(x)$ at $x = c$ by generating tables of values of $f(x)$ for values of x near c.

Hence the numerator and denominator in (4) are both small, nonzero numbers. This means that when t is close to 3, the quotient in (4) is a ratio of two small, nonzero numbers. A problem arises because we cannot say for certain what happens when we divide one small number by another. The result might be relatively small, relatively large, or somewhere in between; for example,

$$0.0001/0.001 = 0.1, \quad 0.001/0.0001 = 10, \quad 0.002/0.001 = 2.$$

Thus simply knowing that the limit in (4) is close to the ratio of two small numbers is not enough information for us to determine what number, if any, is the limit. See Exercises 23, 24, and 25.

TABLE 1.7

t	$\dfrac{2t^2 - 4t - 6}{t^2 - 9}$
2.5	1.27273
3.5	1.38462
2.9	1.32203
3.1	1.34426
2.99	1.33222
3.01	1.33444
2.9999	1.33332
3.0001	1.33334

NUMERICAL APPROACH

With a calculator or computer, we can generate a list of values of

$$\frac{2t^2 - 4t - 6}{t^2 - 9} \tag{5}$$

for t values close to 3. A list of such values is shown in Table 1.7. (You should verify some of the values in this table, and add one or two new lines to the table.) The numerical data seem to indicate that the value of the quotient in (5) gets close to 1.3333 as t nears 3. Thus we have evidence that

$$\lim_{t \to 3} \frac{2t^2 - 4t - 6}{t^2 - 9} \approx 1.333.$$

GRAPHICAL APPROACH

Figure 1.44 shows the graph of

$$y = \frac{2t^2 - 4t - 6}{t^2 - 9}$$

for $2 < t < 3$ and $3 < t < 4$. (You should produce a similar graph on your computer or calculator.) Let t be close to 3 on the horizontal axis. Move vertically from t to the graph and from the graph over to the vertical axis. We meet the vertical axis at a point close to 1.33. This means that when t is close to 3, the value of (5) is close to 1.33. This suggests that

$$\lim_{t \to 3} \frac{2t^2 - 4t - 6}{t^2 - 9} \approx 1.33.$$

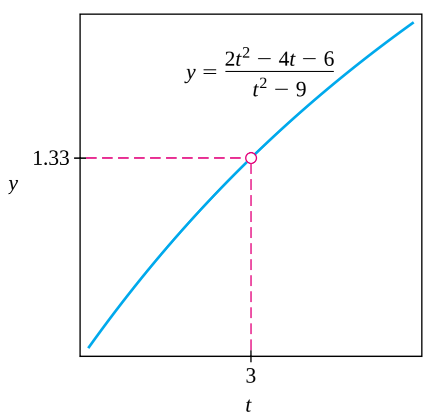

FIGURE 1.44 The graph indicates that $\lim_{t \to 3} \dfrac{2t^2 - 4t - 6}{t^2 - 9} \approx 1.3.$

To get a better approximation of the limit, zoom in on the graph and investigate the graph for t values even closer to 3.

ANALYTICAL APPROACH

After simplifying (5) algebraically, we can get very precise information about the limit. If t is close to 3 (but $t \neq 3$), then the expression in (5) can be simplified:

$$\frac{2t^2 - 4t - 6}{t^2 - 9} = \frac{2(t-3)(t+1)}{(t-3)(t+3)} = \frac{2(t+1)}{t+3}. \tag{6}$$

By canceling the factor of $t - 3$, we have eliminated the factor that causes the numerator and denominator of (5) to approach 0 as t approaches 3. When t is close to 3, the numerator of the last expression is close to $2(3 + 1) = 8$, and the denominator is close to $(3 + 3) = 6$. Thus when t is close to 3, the expression in (6) is close to

$$\frac{2(3+1)}{(3+3)} = \frac{4}{3}.$$

This means that

$$\lim_{t \to 3} \frac{2t^2 - 4t - 6}{t^2 - 9} = \lim_{t \to 3} \frac{2(t+1)}{(t+3)} = \frac{4}{3} \approx 1.33333. \tag{7}$$

Limits and the Rate of Change

In Section 1.4 we worked with the rate-of-change algorithm. This algorithm gives guidelines for determining the rate of change of a function f at a point a. We also saw that the rate-of-change algorithm can be used to identify instances in which the rate of change does not exist. According to the algorithm, the rate of change of f at a is determined by the behavior of the ratio, or "difference quotient"

$$\frac{f(a+h) - f(a)}{h}, \tag{8}$$

for values of h close to 0. Hence the rate of change, if it exists, is the limit of (8) as h tends to 0. We restate the rate-of-change algorithm in this more compact form.

The Rate-of-Change Algorithm

Let f be a function and a a real number in the domain of f. If the rate of change of f at a exists, then it is equal to

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{9}$$

EXAMPLE 4 Calculating a rate of change

Let $y = f(x) = 2x^2 - 3x + 1$. Find the rate of change of y with respect to x at $x = -2$.

Solution

Using (9) with $a = -2$, the rate of change is

$$\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0}\frac{(2(-2+h)^2-3(-2+h)+1)-15}{h}$$
$$= \lim_{h\to 0}\frac{-11h+2h^2}{h}$$
$$= \lim_{h\to 0}(-11+2h) = -11.$$

EXAMPLE 5 Population growth rate

A population grows in such a way that at time t the population is

$$P = P(t) = 3\cdot 2^{t/40}, \tag{10}$$

where t is measured in years and population in billions of individuals. Find the rate of change of population with respect to time at time $t = 50$.

Solution

Using (9) with $a = 50$, the rate of change at time $t = 50$ is

$$\lim_{h\to 0}\frac{P(50+h)-P(50)}{h} = \lim_{h\to 0}\frac{3\cdot 2^{(50+h)/40}-3\cdot 2^{50/40}}{h}$$
$$= \lim_{h\to 0} 3\cdot\frac{2^{50/40}2^{h/40}-2^{50/40}}{h}$$
$$= \lim_{h\to 0} 3\cdot 2^{50/40}\frac{2^{h/40}-1}{h}.$$

Noting that the factor $3\cdot 2^{50/40}$ is $P(50)$, we have

$$\lim_{h\to 0}\frac{P(50+h)-P(50)}{h} = \lim_{h\to 0} P(50)\frac{2^{h/40}-1}{h}.$$

We approximate this last limit numerically by calculating the value of

$$P(50)\frac{2^{h/40}-1}{h}$$

for several values of h close to 0. These calculations are shown in Table 1.8. The values in the right column seem to be close to

$$P(50)\cdot 0.017329 \approx 0.123647. \tag{11}$$

Because the units for t are years and the units for P are billions of individuals, the rate of change has units of billions of people/year. Hence for $t = 50$, the rate of change of population is approximated by

0.123647 billion individuals per year.

This means that in year $t = 50$, the population grows by about 123,647,000 individuals. This is about 1.7 percent of the year 50 population.

TABLE 1.8

h	$P(50)\dfrac{2^{h/40}-1}{h}$
0.1	$0.017344\cdot P(50)$
−0.1	$0.017314\cdot P(50)$
0.005	$0.017329\cdot P(50)$
−0.005	$0.017328\cdot P(50)$
0.0001	$0.017329\cdot P(50)$
−0.0001	$0.017329\cdot P(50)$

Working with Limits

Sums, Products, and Quotients Many of the limits that we work with can be evaluated by algebraically combining simpler limits. For example, in (7) we evaluated

$$\lim_{t\to 3} \frac{2(t+1)}{(t+3)}$$

by noting that the numerator has limit 8 and the denominator has limit 6, and then reasoning that the quotient must have limit 8/6. This is an application of (15) in the following list of rules for limits.

Rules for Combining Limits

Let g and h be two functions with

$$\lim_{x\to a} g(x) = L \quad \text{and} \quad \lim_{x\to a} h(x) = M,$$

and let c be a real number. Then,

$$\lim_{x\to a}(c\cdot g(x)) = c\Big(\lim_{x\to a} g(x)\Big) = cL, \tag{12}$$

$$\lim_{x\to a}(g(x)+h(x)) = \Big(\lim_{x\to a} g(x)\Big) + \Big(\lim_{x\to a} h(x)\Big) = L + M, \tag{13}$$

$$\lim_{x\to a}(g(x)h(x)) = \Big(\lim_{x\to a} g(x)\Big)\Big(\lim_{x\to a} h(x)\Big) = LM. \tag{14}$$

If $M \neq 0$, then

$$\lim_{x\to a} \frac{g(x)}{h(x)} = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{L}{M}. \tag{15}$$

Although these results can all be proved, we will accept these statements without proof because they seem to follow from common sense. For example, to explain (14), we can reason as follows:

When x is close to a, $f(x)$ is close to L and $g(x)$ is close to M. Thus when x is close to a, $f(x)\cdot g(x)$ is close to $L\cdot M$. This suggests that

$$\lim_{x\to a}(g(x)h(x)) = LM.$$

Those who are interested in careful proofs for the limit rules should see Exercises 27 and 28 in Section 1.6.

Polynomials and Trigonometric Functions To use the limit rules effectively, we must be able to break complicated limits into simple ones and then evaluate the simple limits. Here are some of the more common "simple limits."

Limits of Polynomials, Sines, and Cosines

Let a be a real number. Then for any polynomial

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$
$$\lim_{x\to a} p(x) = p(a). \tag{16}$$

Also, for any real a,

$$\lim_{x\to a} \sin x = \sin a \quad \text{and} \quad \lim_{x\to a} \cos x = \cos a. \tag{17}$$

We also accept these results without proof because they seem reasonable given our experience with polynomials and trigonometric functions. We have already applied (16) and (17) in working Examples 1 and 2. Can you see where? When we apply the rules for combining limits and for finding limits of polynomials, sines, and cosines, we usually do not mention the application. However, we give one example to illustrate how we might document uses of these rules.

EXAMPLE 6 Evaluate

$$\lim_{t\to -1} (t^2 - 2t + 7) \tan t. \tag{18}$$

Solution

First recall that $\tan t = \sin t/\cos t$. By (17),

$$\lim_{t\to -1} \sin t = \sin(-1) \quad \text{and} \quad \lim_{t\to -1} \cos t = \cos(-1).$$

Because $\cos(-1) \neq 0$, it follows from (15) that

$$\lim_{t\to -1} \tan t = \lim_{t\to -1} \frac{\sin t}{\cos t} = \frac{\lim_{t\to -1} \sin t}{\lim_{t\to -1} \cos t} = \frac{\sin(-1)}{\cos(-1)} = \tan(-1).$$

Next, because $t^2 - 2t + 7$ is a polynomial, we use (16) to see that

$$\lim_{t\to -1} (t^2 - 2t + 7) = (-1)^2 - 2(-1) + 7 = 10.$$

We now know that the limit of each factor of (18) exists, so we apply (14) to get

$$\begin{aligned}\lim_{t\to -1} (t^2 - 2t + 7) \tan t &= \Big(\lim_{t\to -1} (t^2 - 2t + 7)\Big)\Big(\lim_{t\to -1} \tan t\Big) \\ &= 10 \tan(-1) \approx -15.5741\end{aligned} \tag{19}$$

The graph of $y = (t^2 - 2t + 7) \tan t$ for t near -1 is shown in Fig. 1.45. From the graph we see that when t is close to -1, the value of $(t^2 - 2t + 7) \tan t$ is close to -15. This is consistent with (19).

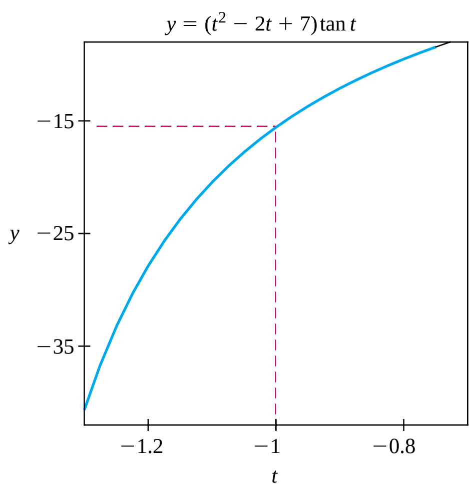

FIGURE 1.45 The graph of $y = (t^2 - 2t + 7) \tan t$ for t near -1.

Compositions With (12), (13), (14), and (15) we can find the limits of functions that are sums, products, or quotients of other functions whose limits we know. Because composition is another important way of combining functions, we give a rule for limits of compositions. This rule seems to be a natural consequence of the definition of limit, so we will accept it without proof. We discuss some aspects of this rule in Exercises 23 and 24 of Section 1.6.

Limits and Compositions

Let h and g be functions. Suppose that

$$\lim_{x\to a} g(x) = L$$

and that h is defined at L and

$$\lim_{x\to L} h(x) = h(L).$$

Then

$$\lim_{x\to a} h(g(x)) = h\Big(\lim_{x\to a} g(x)\Big) = h(L). \tag{20}$$

Most uses of (20) seem consistent with common sense, so we usually do not mention applications of the result. However, the following example is one in which we do point out the use of (20).

EXAMPLE 7 Find the value of

$$\lim_{\theta \to 1/2} \sin(\theta^2).$$

Solution

Let

$$p(\theta) = \theta^2.$$

Because $p(\theta)$ is a polynomial, we can apply (16) to obtain

$$\lim_{\theta \to 1/2} p(\theta) = p(1/2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Next note that $\sin t$ is defined for $t = 1/4$ and that by (17)

$$\lim_{t \to 1/4} \sin t = \sin\left(\frac{1}{4}\right).$$

Now apply (20) to get

$$\begin{aligned} \lim_{\theta \to 1/2} \sin(\theta^2) &= \lim_{\theta \to 1/2} \sin(p(\theta)) \\ &= \sin\left(\lim_{\theta \to 1/2} p(\theta)\right) = \sin\left(\frac{1}{4}\right) \approx 0.247404 \end{aligned} \quad (21)$$

The graph of $y = \sin(\theta^2)$ for θ near 1/2 is shown in Fig. 1.46. Is the answer in (21) consistent with the graph?

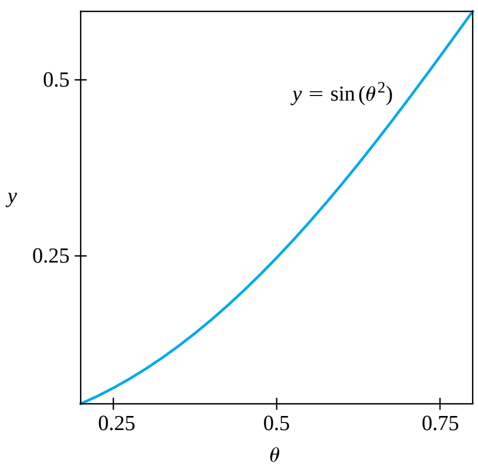

FIGURE 1.46 The graph of $y = \sin(\theta^2)$ for θ near 1/2.

A Trigonometric Limit

In Chapter 2 we will calculate the rates of change of many different functions. When we find the rates of change of the sine and cosine functions, we will need to know the value of

$$\lim_{h \to 0} \frac{\sin h}{h}.$$

We evaluate this limit in the next example. In doing so, we see more techniques for working with limit and to review some trigonometry. For a more complete review, see Section A.2 in the Appendix.

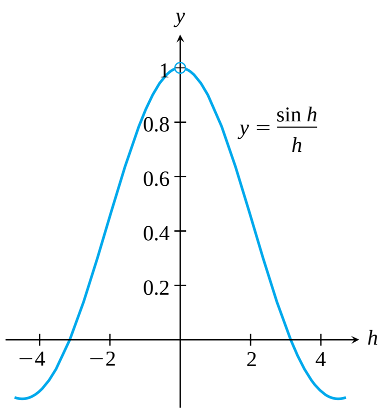

FIGURE 1.47 The graph of $y = (\sin h)/h$ suggests that $\lim_{h \to 0}(\sin h)/h = 1$.

EXAMPLE 8 Evaluate $\lim_{h \to 0} \frac{\sin h}{h}$, where h is in radians.

Solution

We first get an estimate for the limit by graphing

$$y = \frac{\sin h}{h} \quad (22)$$

for h values close to 0. The graph is shown in Fig. 1.47 and illustrates that the expression in (22) gets close to 1 as h gets close to 0. Hence we guess that

$$\lim_{h \to 0} \frac{\sin h}{h} \approx 1.$$

We could also have found this estimate by gathering numerical data. See Exercise 13.

To verify analytically that the value of the limit is 1, we first show that for h close to 0,

$$\cos h \le \frac{\sin h}{h} \le \frac{1}{\cos h}. \tag{23}$$

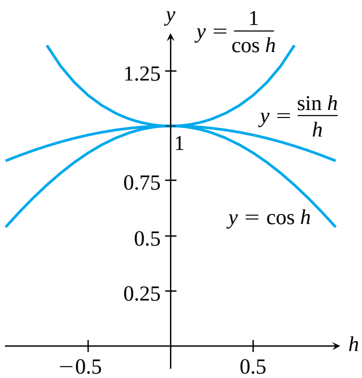

FIGURE 1.48 When h is small, $\sin h/h$ is between $\cos h$ and $1/\cos h$.

In Fig. 1.48 we show the graphs of

$$y = \cos h, \quad y = \frac{\sin h}{h}, \quad \text{and} \quad y = \frac{1}{\cos h},$$

and note that this figure does suggest (23). We start by establishing this inequality for small, positive values of h. Referring to Fig. 1.49, consider the circle of center $O = (0, 0)$ and radius 1. Let $S = (1, 0)$ and label P on the upper half of the circle so that $\angle POS$ has (radian) measure h. Then

$$P = (\cos h, \sin h).$$

From P draw a segment perpendicular to the x-axis. This segment intersects the axis at $R = (\cos h, 0)$. Let the line perpendicular to the x-axis at S meet OP at Q. Because triangle OSQ is a right triangle with right angle at S and $OS = 1$, it follows that

$$QS = \frac{QS}{1} = \frac{QS}{OS} = \frac{\text{opposite}}{\text{adjacent}} = \tan h.$$

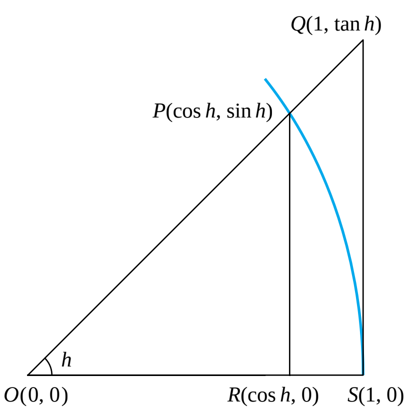

FIGURE 1.49 A right triangle in the circle of center $(0, 0)$ and radius 1.

From Fig. 1.49,

$$\text{area}(\triangle ORP) < \text{area}(\text{sector } OSP) < \text{area}(\triangle OSQ). \tag{24}$$

Sector OSP has central angle h radians. The area of sector OSP is

$$\text{area}(\text{sector } OSP) = \frac{1}{2} 1^2 \cdot h = \frac{1}{2} h. \tag{25}$$

(See Section A.2 in the Appendix, where it is proved that in a circle of radius a, the area of asector with central angle θ is $\frac{1}{2}a^2\theta$.) In addition,

$$\text{area}(\triangle ORP) = \frac{1}{2}\cos h \sin h \quad \text{and} \quad \text{area}(\triangle OSQ) = \frac{1}{2}\tan h. \tag{26}$$

Combining (24), (25), and (26), we have

$$\frac{1}{2}\cos h \sin h \le \frac{1}{2} h \le \frac{1}{2}\frac{\sin h}{\cos h} \tag{27}$$

Because we have assumed that h is a small, positive number, $\sin h > 0$. Multiplying inequality (27) by $2/\sin h$ gives

$$\cos h \le \frac{h}{\sin h} \le \frac{1}{\cos h}. \tag{28}$$

We now take reciprocals of the expressions in (28), recalling that if $0 < a < b$, then $0 < 1/b < 1/a$. We obtain

$$\cos h \le \frac{\sin h}{h} \le \frac{1}{\cos h}.$$

This establishes (23) for h positive and close to 0. In Exercise 44 we ask you to show that inequality is also true if h is small and negative. Hence (23) is true for all small, nonzero h.

We now use (23) to show that $\lim_{h\to 0}\frac{\sin h}{h} = 1$. First observe that

$$\lim_{h\to 0}\cos h = \cos 0 = 1 \quad \text{and} \quad \lim_{h\to 0}\frac{1}{\cos h} = \frac{1}{\lim_{h\to 0}\cos h} = 1.$$

From (23) we see that for small, nonzero h, the expression $(\sin h)/h$ is "trapped" between $\cos h$ and $1/\cos h$. See Fig. 1.48. Since $\cos h$ and $1/\cos h$ both tend to 1 as h approaches 0, it must be true that $(\sin h)/h$ also approaches 1 as h tends to 0, that is

$$\lim_{h\to 0}\frac{\sin h}{h} = 1.$$

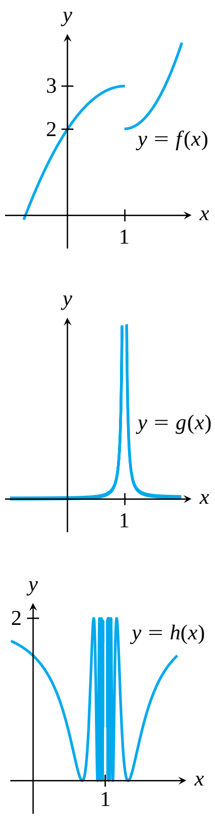

FIGURE 1.50 Sometimes a limit does not exist.

Existence of Limits

Figure 1.50 shows the graphs of three functions, f, g, and h, near $x = 1$. The graphs of these three functions look very different near $x = 1$. From the graph of f we see that if $x < 1$ and close to 1, then $f(x)$ is close to 3, but if $x > 1$ and close to 1, then $f(x)$ is close to 2.

From the graph of g we see that as x gets close to 1, the values of $g(x)$ get bigger and bigger. They seem to be growing to infinity, so the line $x = 1$ may be an asymptote to the graph of g.

The graph of h oscillates wildly between the horizontal lines $y = 0$ and $y = 2$ as x gets close to 1.

Though these graphs look very different, they do have one thing in common. Each graph indicates that *as x gets close to 1, the function values do not approach one and only one finite real number.* More specifically, as x approaches 1, $f(x)$ gets close to two different numbers, $g(x)$ does not get close to any finite number, and $h(x)$, as it oscillates, gets close to every number between 0 and 2. For each of these three functions, the limit as x approaches 1 does not exist.

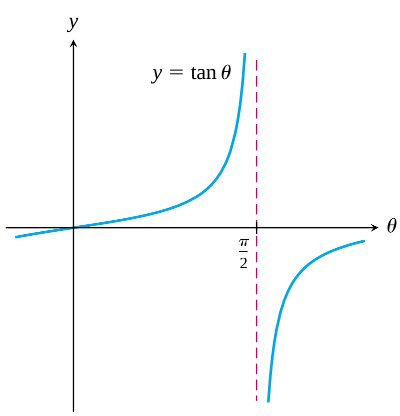

FIGURE 1.51 The graph indicates that $\lim_{\theta\to\pi/2}\tan\theta$ does not exist.

When Limits Do Not Exist

Let F be a function and a a real number. If there is no single finite number L such that $F(x)$ gets close to L (and only L) as x gets close to a, then

$$\lim_{x\to a}F(x) \text{ does not exist.}$$

EXAMPLE 9 Show that $\lim_{\theta\to\pi/2}\tan\theta$ does not exist.

Solution

When evaluating a limit of a function, it is often a good idea to look at a graph of the function. In Fig. 1.51 we see the graph of $y = \tan\theta$ for θ near

TABLE 1.9

θ	$\tan\theta$
$\pi/2 + 0.5$	-1.83049
$\pi/2 - 0.5$	1.83049
$\pi/2 + 0.1$	-9.96664
$\pi/2 - 0.1$	9.96664
$\pi/2 + 0.01$	-99.9967
$\pi/2 - 0.01$	99.9967
$\pi/2 + 0.001$	-1000.0
$\pi/2 - 0.001$	1000.07

$\pi/2$. The graph suggests that the line $\theta = \pi/2$ is a vertical asymptote to the graph. When $\theta > \pi/2$ and close to $\pi/2$, the values of $\tan\theta$ are "large and negative." (This means that $\tan\theta < 0$ and $|\tan\theta|$ is large.) Thus it appears that the function values are heading toward $-\infty$ as θ closes in on $\pi/2$ from the right. When $\theta < \pi/2$ and close to $\pi/2$, the values of $\tan\theta$ are large and positive, and it appears that the function values are approaching $+\infty$ as θ approaches $\pi/2$ from the left. Since $\tan\theta$ is not getting close to a single number as θ gets close to $\pi/2$, we conclude that

$$\lim_{\theta\to\pi/2} \tan\theta \text{ does not exist.}$$

We can also investigate the behavior of the tangent function near $\theta = \pi/2$ by computing $\tan\theta$ for several θ values near $\pi/2 \approx 1.5708$. See Table 1.9. The data in the table reflect the behavior of the graph and so also suggest that

$$\lim_{\theta\to\pi/2} \tan\theta \text{ does not exist.}$$

Exercises 1.5

Exercises 1–12: Determine the limit, if it exists. Use analytical methods, then check your answer using graphical or numerical methods.

1. $\lim_{x\to 2}(x^2 - 3x)/(x + 1)$

2. $\lim_{t\to -1/2} \sin \pi t$

3. $\lim_{h\to 0}(3h^3 - 7h^2 + h)/h$

4. $\lim_{x\to -4}(2x^2 + 7x - 4)/(x + 4)$

5. $\lim_{t\to 1/2} \dfrac{2t + 1}{2t - 1}$

6. $\lim_{r\to\sqrt{2}} \dfrac{r^2 - 2}{r^3 + 2r^2 - 2r - 4}$

7. $\lim_{h\to 0} \dfrac{g(4 + h) - g(4)}{h}$, where $g(t) = t^3 - 3t$

8. $\lim_{h\to 0} \dfrac{G(-8 + h) - G(-8)}{h}$, where $G(s) = 2/(s + 1)$

9. $\lim_{x\to -1/3} \dfrac{\sqrt{6x + 3} - 1}{6x + 2}$

10. $\lim_{x\to -1} f(x)$, where

$$f(x) = \begin{cases} x + 1 & (x < -1) \\ 2 - 2x^2 & (-1 \le x \le 2) \\ 0 & (x > 2) \end{cases}$$

11. $\lim_{x\to 2} f(x)$ for f as given in Exercise 10.

12. $\lim_{h\to 0} \dfrac{f(-1 + h) - f(-1)}{h}$ for f as given in Exercise 10.

T **13.** Use numerical techniques to investigate

$$\lim_{h\to 0} \frac{\sin h}{h}.$$

T **14.** Investigate

$$\lim_{\theta\to 0} \frac{1 - \cos\theta}{\theta^2}.$$

Use graphical and numerical techniques.

T **15.** Investigate

$$\lim_{\theta\to 0} \frac{\tan\theta}{\theta}.$$

Use graphical and numerical techniques.

T **16.** Investigate

$$\lim_{\theta\to 0} \cos\left(\frac{1}{\theta}\right).$$

Use graphical and numerical techniques.

T **17.** Investigate

$$\lim_{\theta\to\pi/4} \frac{\tan\theta - 1}{\theta - \pi/4}.$$

Use graphical and numerical techniques.

18. The accompanying figure shows the graph of a function f.

a. For what values of c does $\lim_{x\to c} f(x) = 1$?

b. For what values of c does $\lim_{x\to c} f(x)$ not exist?

c. Evaluate $\lim_{x\to -1} f(x)$.

d. Evaluate $\lim_{x\to -1} f(x^2)$.

e. Evaluate $\lim_{x\to -1} f(\sqrt{x})$.

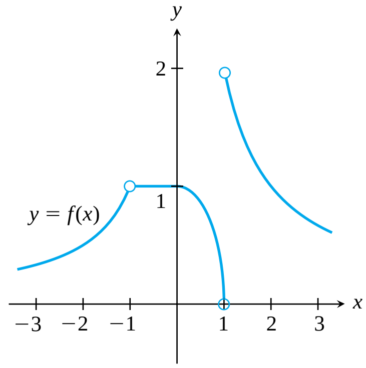

Figure for Exercise 18.

19. The accompanying figure shows the graph of a function g.

a. For what values of c does $\lim_{x\to c} g(x)$ fail to exist?

b. Evaluate $\lim_{x\to 2} g(x)$.

c. Evaluate $\lim_{x\to 2} (g(x))^2$.

d. Evaluate $\lim_{x\to 2} (g(x))^3$.

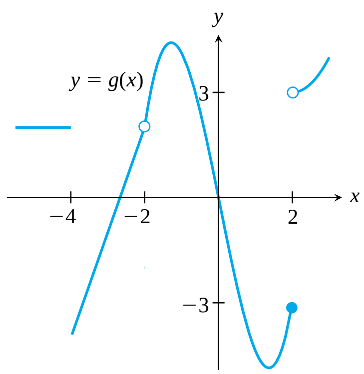

Figure for Exercise 19.

20. From the graph in Fig. 1.40, what is the temperature as the average molecular speed approaches 0?

T **21.** Investigate the following limits both graphically and numerically:

a. $\lim_{h\to 0} \dfrac{2^h - 1}{h}$

b. $\lim_{h\to 0} \dfrac{3^h - 1}{h}$

c. $\lim_{h\to 0} \dfrac{(1/2)^h - 1}{h}$

d. $\lim_{h\to 0} \dfrac{(1/3)^h - 1}{h}$

T **22.** Let a be a fixed positive number. Consider the limit

$$\lim_{h\to 0} \frac{a^h - 1}{h}.$$

By using numerical methods, find an approximate value of a for which this limit is 1.

23. Find real numbers a and b with $-0.01 \le a, b \le 0.01$ and:

a. $a/b = 3$.

b. $a/b = -500$.

c. $a/b = 10^9$.

d. $a/b = -10^{-9}$.

24. Let a and b be two real numbers. Find the largest and smallest possible values of a/b if:

a. $3.9 \le a \le 4.1$ and $5 \le b \le 5.2$.

b. $-3.1 \le a \le -2.9$ and $5 \le b \le 5.2$.

c. $-0.1 \le a \le 0.1$ and $9.9 \le b \le 10.2$.

25. Find functions f and g such that

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x) = 0$$

and:

a. $\lim_{x\to 3} \dfrac{f(x)}{g(x)} = 5$.

b. $\lim_{x\to 3} \dfrac{f(x)}{g(x)} = -\sqrt{3}$.

c. $\lim_{x\to 3} \dfrac{f(x)}{g(x)}$ does not exist.

26. Let $0 < c < \pi/2$ and $0 < d < \pi/2$. If $\lim_{\theta\to c} \sin\,\theta = 0.3$ and $\lim_{\theta\to d} \sin\,\theta = 0.9$, find:

a. $\lim_{\theta\to c+d} \sin\,\theta$.

b. $\lim_{\theta\to c-d} \cos\,\theta$.

c. $\lim_{\theta\to 2d} \cos\,\theta$.

27. The graph of a function f is shown in the accompanying figure. On the same set of axes draw the graph of a function g so that $\lim_{x\to -2}(f(x) + g(x))$ does not exist, and $\lim_{x\to 0}(f(x) + g(x)) = 1$.

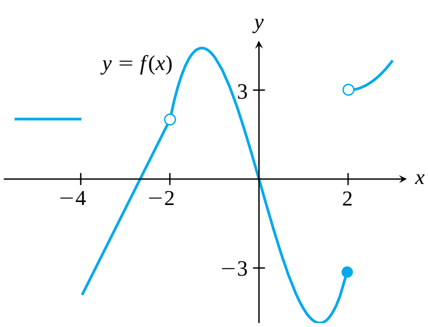

Figure for Exercise 27.

28. Let f be a function and assume that $\lim_{x\to a} f(x) = L$. Is it then true that

$$\lim_{x\to a} |f(x) - L| = 0?$$

Give reasons for your answer.

Exercises 29–36: Find the rate of change of y with respect to x for x = a.

29. $y = -2x + 3, \quad a = 4$

30. $y = x^2 - 3x, \quad a = -1$

31. $y = \dfrac{1}{x+1}, \quad a = 1$

32. $y = x^3, \quad a = -2$

33. $y = x^3, \quad a = 0$

34. $y = -3x^2 + 4x + 2, \quad a = -1$

35. $y = \dfrac{2}{x-2}, \quad a = 0$

36. $y = \dfrac{x}{x+1}, \quad a = 2$

37. Let g be a function and assume that $\lim_{x\to a}|g(x) - M| = 0$. What is the value of

$$\lim_{x\to a} g(x)?$$

Give reasons for your answer.

T **38.** Let $g(t) = t^3 - 4t - 4$.

- **a.** Verify that $\lim_{t\to -2} g(t) = -4$.
- **b.** Answer the following question by looking at a graph: If t is within 0.01 of -2, then what is the largest $|g(t) - (-4)|$ can be?
- **c.** By looking at a graph, find a positive number r that satisfies the following: If t is within r of -2, then $|g(t) - (-4)| < 0.05$.

T **39.** Let $f(x) = x^2 + 2x + 2$.

- **a.** Verify that $\lim_{x\to 1} f(x) = 5$.
- **b.** Answer the following question by looking at a graph: If x is within 0.05 of 1, then what is the largest $|f(x) - 5|$ can be?
- **c.** By looking at a graph, find a positive number r that satisfies the following: If x is within r of 1, then $|f(x) - 5| < 0.1$.

T **40.** Let $f(x) = \left(\sqrt{x} - 2\right)/(x - 4)$.

- **a.** Verify that $\lim_{x\to 4} f(x) = 1/4$.
- **b.** Answer the following question by looking at a graph: If x is within 0.05 of 4, then what is the largest $|f(x) - 1/4|$ can be?
- **c.** By looking at a graph, find a positive number r that satisfies the following: If x is within r of 4, then $|f(x) - 1/4| < 0.1$.

T **41.** Let $r(\theta) = (\sin 2\theta)/\theta$.

- **a.** Verify that $\lim_{\theta\to 0} r(\theta) = 2$.
- **b.** Answer the following question by looking at a graph: If θ is within 0.1 of 0, then what is the largest $|r(\theta) - 2|$ can be?
- **c.** By looking at a graph, find a positive number r that satisfies the following: If θ is within r of 0, then $|r(\theta) - 2| < 0.05$.

42. Charles' law The accompanying table shows how the volume of a gas varies with temperature when the pressure of the gas is kept constant. Plot these data, and then find a line approximating the data. With the aid of this line, find the limit of the temperature as the volume approaches 0. (These data were generated by computer as part of a chemistry lab to investigate methods of finding the Celsius temperature of absolute zero.)

Volume (cm^3)	Temperature (°C)
230	9.4
233.9	14.3
238.1	19.4
242.9	25.3
246.7	30.0
250.9	35.1

Table for Exercise 42.

43. Use Fig. 1.49 to give a geometric argument that $|\sin h| \le |h|$.

44. Show that (23) also holds when h is a small negative number. (*Hint:* Recall that $\sin(-h) = -\sin h$ for all real numbers h.)

45. In Example 8 we found the limit of $(\sin h)/h$ as h approaches 0 by trapping $(\sin h)/h$ between two functions with known limits. In doing so, we illustrated an application of the squeeze theorem:

Let f, g, and h be three functions, each defined for x values close to a but possibly not defined at $x = a$. Assume that

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

and that for x near a,

$$f(x) \le g(x) \le h(x).$$

Then

$$\lim_{h\to a} g(x) = L.$$

- **a.** Draw a good picture illustrating the squeeze theorem.
- **b.** Write a paragraph explaining why the squeeze theorem is true.
- **c.** Explain how the squeeze theorem was used in Example 8.

46. With the aid of the accompanying figure, give a geometric argument that:

a. $|\sin\theta - \sin\alpha| \le |\theta - \alpha|$.

b. $|\cos\theta - \cos\alpha| \le |\theta - \alpha|$.

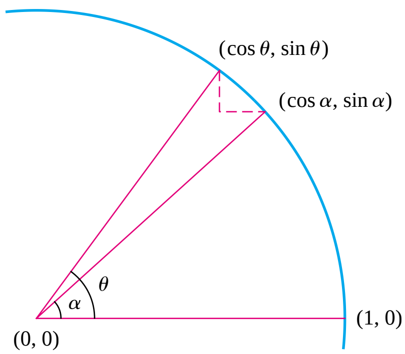

Figure for Exercise 46.

47. Use the squeeze theorem of Exercise 45 and the inequalities developed in Exercise 46 to show that:

a. $\lim_{\theta\to\alpha} \sin\theta = \sin\alpha$.

b. $\lim_{\theta\to\alpha} \cos\theta = \cos\alpha$.

How does this relate to (17)?

48. The world-record mile The accompanying table shows the history of the world record for the 1-mile run since 1875. These data are plotted in the accompanying figure.

a. Use any method to extrapolate the mile-run records to the year 2010.

b. Calculate the average of the times and the average of the dates using the table. Add the point for the average year and average time to the graph. (Such a point is sometimes called a *centroid*.) Put a nail through the centroid. Place a ruler against the nail and try to position the ruler so that if you were to draw a line with the ruler in this position, the line would "best fit" the data. Use this best-fit line to estimate what the record will be in 2010.

c. From these data, does it appear that there is a "limit" to how fast humans can run the mile? Give reasons for your answer, and discuss what you mean by "limit" in this situation.

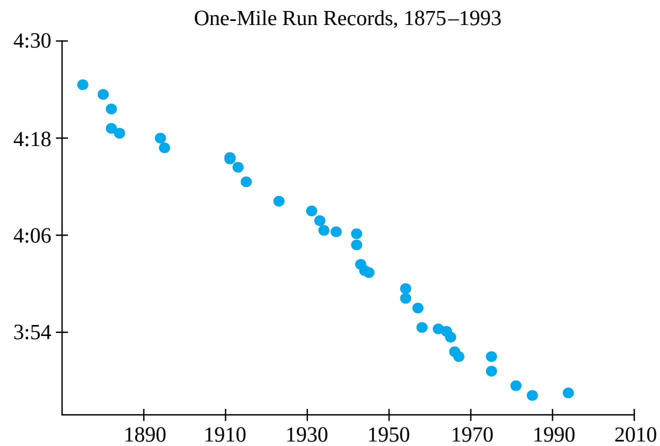

Figure for Exercise 48.

Year	Name (Country)	Time	Year	Name (Country)	Time
1875	Walter Slade (Britain)	4:24.5	1942	Gunder Haegg (Sweden)	4:04.6
1880	Walter George (Britain)	4:23.2	1943	Arne Andersson (Sweden)	4:02.6
1882	Walter George (Britain)	4:21.4	1944	Arne Andersson (Sweden)	4:01.6
1882	Walter George (Britain)	4:19.4	1945	Gunder Haegg (Sweden)	4:01.4
1884	Walter George (Britain)	4:18.4	1954	Roger Bannister (Britain)	3:59.4
1894	Fred Bacon (Scotland)	4:18.2	1954	John Landy (Australia)	3:58.0
1895	Fred Bacon (Scotland)	4:17.0	1957	Derek Ibbotson (Britain)	3:57.2
1911	Thomas Connett (U.S.)	4:15.6	1958	Herb Elliot (Australia)	3:54.5
1911	John Paul Jones (U.S.)	4:15.4	1962	Peter Snell (New Zealand)	3:54.4
1913	John Paul Jones (U.S.)	4:14.6	1964	Peter Snell (New Zealand)	3:54.1
1915	Norman Tauber (U.S.)	4:12.6	1965	Michel Jazy (France)	3:53.6
1923	Paavo Nurmi (Finland)	4:10.4	1966	Jim Ryun (U.S.)	3:51.3
1931	Jules Ladoumegue (France)	4:09.2	1967	Jim Ryun (U.S.)	3:51.1
1933	Jack Lovelock (New Zealand)	4:07.6	1975	Filbert Bayi (Tanzania)	3:51.0
1934	Glenn Cunningham (U.S.)	4:06.8	1975	John Walker (New Zealand)	3:49.4
1937	Sydney Wooderson (Britain)	4:06.4	1981	Sebastian Coe (Britain)	3:47.33
1942	Gunder Haegg (Sweden)	4:06.2	1985	Steve Cram (Britain)	3:46.30
1942	Arne Andersson (Sweden)	4:06.2	1993	Noureddine Morceli (Algeria)	3:44.39

Table for Exercise 48.