5.3 The Fundamental Theorem of Calculus

In Section 5.2 we defined the definite integral $\int_a^b f(x)\,dx$ and interpreted this number as the net area of the region $\mathcal{R}$ between the graph of f and the x-axis. Figure 5.14(a) shows a representative function, interval, and region. We defined the integral as the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

of Riemann sums. We were able to evaluate each of several integrals by first summing the associated Riemann sum R_n with the help of a formula and then taking the limit as $n \to \infty$ of the resulting expression. This process—summing by formula and then taking the limit—is feasible only for relatively simple functions.

In this section we study the Fundamental Theorem of Calculus, which greatly simplifies the evaluation of $\int_a^b f(x)\,dx$ for many functions. We use the context of motion to give a heuristic argument for the Fundamental Theorem of Calculus. For this reason, we use the notation $\int_a^b v(t)\,dt$ instead of $\int_a^b f(x)\,dx$, where $v(t)$ is the velocity of a car moving along a straight highway during the time interval $[a, b]$. Figure 5.14 shows graphs of identical functions $y = f(x)$ and $v = v(t)$ on $[a, b]$. Part (a) of the figure interprets $\int_a^b f(x)\,dx$ in terms of the area of a region $\mathcal{R}$. We use part (b) to interpret $\int_a^b v(t)\,dt$ in terms of motion.

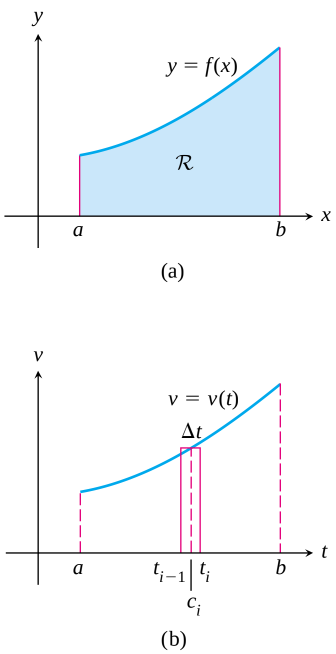

FIGURE 5.14 (a) The area of the shaded region $\mathcal{R}$ is $\int_a^b f(x)\,dx$. (b) Interpreting the integral $\int_a^b v(t)\,dt$ in terms of motion.

We assume that distances are measured in kilometers, time in hours, and velocity in kilometers per hour (kph). We argue that the area of the region beneath the velocity graph is equal to the distance traveled by the object. We lay out the main steps in the argument first and then discuss them in more detail.

Step 1: We already know that $\int_a^b v(t)\,dt$ is equal to the area A of the region $\mathcal{R}$ beneath the velocity graph. This is only a change of notation from Fig. 5.14(a). From the definition of the integral we know that

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} v(c_i)\,\Delta t = \int_a^b v(t)\,dt, \tag{1}$$

where $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ is an evaluation set for the regular subdivision $\mathcal{S}_n$ of $[a, b]$.

Step 2: Referring to Fig. 5.14(b), if for each $i = 1, \ldots, n$ we can show that there is a time c_i in the subinterval $[t_{i-1}, t_i]$ so that the number $v(c_i)\,\Delta t$ is the *exact* distance traveled by the car during this subinterval, then the Riemann sum $R_n = \sum_{i=1}^n v(c_i)\,\Delta t$ is *exactly* equal to the distance traveled by the car during the time interval $[a, b]$.

Step 3: Letting $x(t)$ be the position of the car at any time $t \in [a, b]$,

$$x(b) - x(a) = \int_a^b v(t)\,dt. \tag{2}$$

The key to the entire argument is in Step 2, to show that there is a time $c_i \in [t_{i-1}, t_i]$ for which the number $v(c_i)\,\Delta t$ is the exact distance traveled by the car during this subinterval. This follows from the Mean-value Theorem. Letting $x(t)$ be the position of the car for all $t \in [t_{i-1}, t_i]$, the Mean-value Theorem (see Section 4.3, page 282) states that there is a number $c_i \in (t_{i-1}, t_i)$ for which

$$x(t_i) - x(t_{i-1}) = x'(c_i)\,(t_i - t_{i-1}). \tag{3}$$

Because $x'(t) = v(t)$ and $x(t_i) - x(t_{i-1})$ is the distance traveled by the car during the subinterval $[t_{i-1}, t_i]$, Equation (3) states that a time c_i can be chosen in the subinterval

$[t_{i-1}, t_i]$ so that the number $v(c_i)\,\Delta t$ is the exact distance traveled by the car during this subinterval. *Note:* We are assuming that the velocity is never negative during the time interval $[a, b]$, which is consistent with the graph of v shown in Fig. 5.14(b).

Because $v(c_i)\,\Delta t$ is the exact distance traveled by the car during the subinterval $[t_{i-1}, t_i]$, it follows that R_n is the exact distance $x(b) - x(a)$ traveled by the car during the entire interval $[a, b]$. This also follows from

$$\begin{aligned}\sum_{i=1}^{n} v(c_i)\,\Delta t &= (x(t_1) - x(t_0)) + (x(t_2) - x(t_1)) + \cdots + (x(t_n) - x(t_{n-1})) \\ &= x(t_n) - x(t_0) = x(b) - x(a).\end{aligned}$$

With the evaluation sets chosen so that (3) is satisfied, each Riemann sum R_n is equal to $x(b) - x(a)$. It follows that $\lim_{n\to\infty} R_n = x(b) - x(a)$. But it is also true that $\lim_{n\to\infty} R_n = A$. Equation (2) now follows from (1).

How can we use this result to evaluate an integral $\int_a^b v(t)\,dt$? Well, if we can find a function $x = x(t)$ for which $x'(t) = v(t)$, then, according to (2),

$$\int_a^b v(t)\,dt = x(b) - x(a).$$

Here's a quick example: Recall that we showed in Example 1 of Section 5.2 that $\int_0^1 x^2\,dx = 1/3$ or, switching the integration variable to t, $\int_0^1 t^2\,dt = 1/3$. We can now calculate this integral more easily. All we must do is find a function $x = x(t)$ whose derivative is $v = v(t) = t^2$. A moment's thought shows that $\left(\frac{1}{3}t^3\right)' = t^2$. Hence, if we take $x(t) = \frac{1}{3}t^3$,

$$\int_0^1 t^2\,dt = x(1) - x(0) = \left(\frac{1}{3}1^3\right) - \left(\frac{1}{3}0^3\right) = \frac{1}{3}.$$

To discuss the Fundamental Theorem of Calculus much further, we need two properties of the integral. Although we assume these without proof, we show that they are reasonable when interpreted in terms of area. We end the section by listing two additional properties of the integral. These properties follow from the Fundamental Theorem of Calculus and are left as exercises. All functions considered in this chapter are assumed to be continuous.

The Fundamental Theorem of Calculus

Figure 5.15 shows the region $\mathcal{R}$ lying beneath the graph of the nonnegative function f. This region can be described more precisely as

$$\mathcal{R} = \{(x, y) : a \le x \le b, \qquad 0 \le y \le f(x)\}.$$

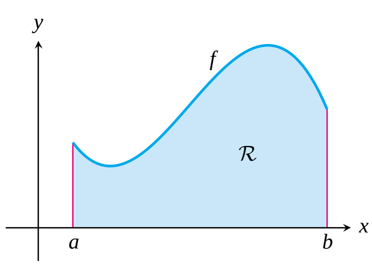

FIGURE 5.15 $\mathcal{R}$ is the region lying beneath the graph of f.

DEFINITION 1 Area and Net Area

Let f be a continuous, nonnegative function on $[a, b]$. The area of the region $\mathcal{R}$ beneath the graph of f is the number $\int_a^b f(x)\,dx$. If we assume only that f is continuous on $[a, b]$, then the net area of the region between f and the segment $[a, b]$ of the x-axis is the number $\int_a^b f(x)\,dx$.

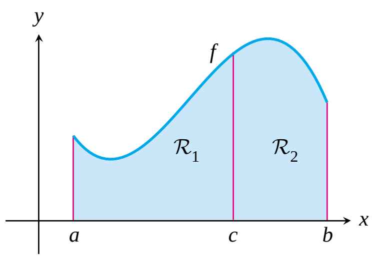

FIGURE 5.16 The area beneath the curve, from a to b, is the sum of the areas of the regions $\mathcal{R}_1$ and $\mathcal{R}_2$.

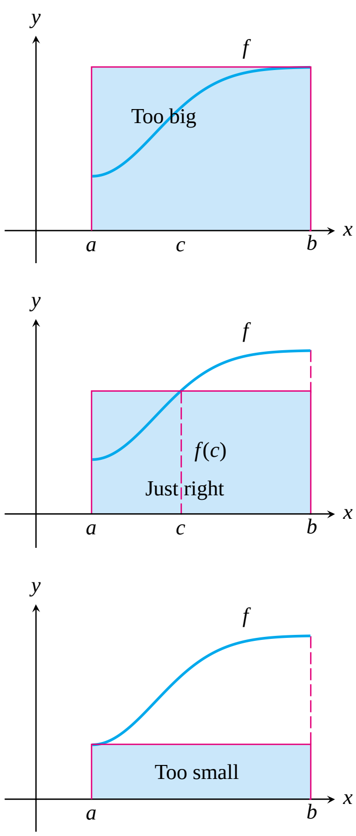

FIGURE 5.17 Mean-value Theorem for Integrals.

DEFINITION 2 Assigning a Value to the Definite Integral $\int_a^b f(x)\,dx$ When $b = a$

For any continuous function f defined on an interval $[a, b]$, we define

$$\int_a^a f(x)\,dx = 0. \tag{4}$$

The intuitive content of Definition 2 is that the net area between the graph of f and the (degenerate) interval $[a, a]$ is 0. Look back at Fig. 5.15 and imagine that the right boundary ($x = b$) is moved towards the left boundary ($x = a$). As $b \longrightarrow a$, the net area of the region $\mathcal{R}$ approaches 0. For $b = a$, the area of $\mathcal{R}$ is 0.

PROPERTY 1: The Integral Is Additive

For any continuous function f defined on an interval $[a, b]$ and any point c in $[a, b]$,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx. \tag{5}$$

If we assume that f is nonnegative on $[a, b]$ and interpret the integral as an area, this property is easy to understand. Referring to Fig. 5.16, the sum of the areas $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ of the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ beneath the graph of f is equal to the area $\int_a^b f(x)\,dx$ of the combined region $\mathcal{R}_1 \cup \mathcal{R}_2$ beneath the graph of f.

PROPERTY 2: Mean-value Theorem for Integrals

For any continuous function f defined on an interval $[a, b]$ there is a point c in $[a, b]$ for which

$$\int_a^b f(x)\,dx = f(c)\,(b - a). \tag{6}$$

If we assume that f is nonnegative on $[a, b]$ and interpret the integral as an area, this property is easy to understand. Referring to Fig. 5.17, Property 2 states that there is an "average value" $f(c)$ of f such that the rectangle of width $b - a$ and height $f(c)$ is equal to the area $\int_a^b f(x)\,dx$ of the region beneath the graph of f. For the increasing function f shown in the figure, think of a "dynamic rectangle," one with constant width $(b - a)$ and heights increasing from $f(a)$ to $f(b)$. The area of the rectangle at the bottom of Fig. 5.17, with height $f(a)$, is too small; the area of the rectangle at the top of the figure, with height $f(b)$, is too big; and, in the middle of the figure, Property 2 says that there is an average value $f(c)$ of f for which the area of the rectangle is "just right," that is, equal to $\int_a^b f(x)\,dx$.

We use additivity and the Mean-value Theorem for Integrals in discussing the Fundamental Theorem of Calculus. We also need to understand integrals of the form

$$\int_a^x f(w)\,dw, \qquad \text{where } a < x < b. \tag{7}$$

Note that in (7) we used w, not x, as the *integration variable.* We did this to avoid confusion with the upper limit of integration. The integration variable plays the same role as an index of summation. Just as we recognize that both of the sums

$$\sum_{i=1}^{3} i^2 \quad \text{and} \quad \sum_{j=1}^{3} j^2$$

are equal to $1^2 + 2^2 + 3^2$, the values of the integrals

$$\int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b f(w)\,dw$$

are equal.

The value of the integral in (7) is a function of x. Denoting this function by F and referring to Fig. 5.18, $F(x) = \int_a^x f(w)\,dw$ is the area of the region beneath the graph of f, between $w = a$ and $w = x$.

For stating the Fundamental Theorem, it is convenient to define the term "antiderivative." We say that a function F is an *antiderivative* of f if $F'(x) = f(x)$ for x in some interval $[a, b]$.

We now have all of the preliminary definitions and properties we need to state and prove the Fundamental Theorem of Calculus.

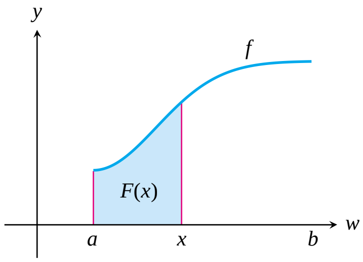

FIGURE 5.18 Area $F(x)$ as a function of x.

THEOREM The Fundamental Theorem of Calculus

Assume that f is continuous on $[a, b]$.

Part I: The function F defined on $[a, b]$ by

$$F(x) = \int_a^x f(w)\,dw, \qquad x \in [a, b], \tag{8}$$

is an antiderivative of f; that is,

$$F'(x) = \frac{d}{dx}\int_a^x f(w)\,dw = f(x), \qquad x \in [a, b]. \tag{9}$$

Part II: If G is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a). \tag{10}$$

PROOF OF PART I By definition, the derivative of F at x is

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h}. \tag{11}$$

Assuming that $h > 0$, it follows directly from Fig. 5.19 that the numerator of this difference quotient is

$$F(x+h) - F(x) = \int_a^{x+h} f(w)\,dw - \int_a^x f(w)\,dw = \int_x^{x+h} f(w)\,dw, \tag{12}$$

because $F(x+h)$ is the area from $w = a$ up to $w = x + h$ and $F(x)$ is the area from $w = a$ to $w = x$. This also follows from Property 1. See Exercise 47. See Exercise 48 for the $h < 0$ case.

Applying the Mean-value Theorem for Integrals to the interval $[x, x+h]$, there is a $c \in [x, x+h]$ for which

$$F(x+h) - F(x) = \int_x^{x+h} f(w)\,dw = f(c)h. \tag{13}$$

Such a c is shown in Fig. 5.19. The area $f(c)h$ of the yellow rectangle is equal to the area of the region between the graph of f and the interval $[x, x+h]$. Hence, from (11),

$$F'(x) = \lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0^+} f(c) = f(x).$$

The last equality holds because f is continuous at x and the number c is trapped between x and $x + h$. Hence, as $h \to 0$, $c \to x$. A similar argument holds in case $h < 0$.

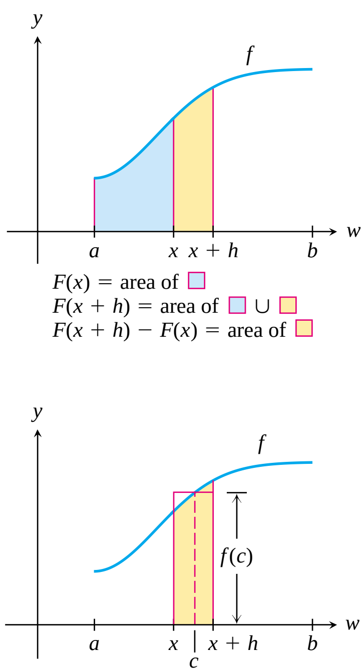

FIGURE 5.19 The regions with areas $F(x)$, $F(x+h)$, and $F(x+h) - F(x)$.

PROOF OF PART II We gave the idea of this proof in the introduction to this section. For each integer n, we choose the evaluation set $\{c_1, c_2, \ldots, c_n\}$ for the regular subdivision $\mathcal{S}_n$ of $[a, b]$ as follows: applying the Mean-value Theorem to the function G on the subinterval $[x_{i-1}, x_i]$, there is a point $c_i \in (x_{i-1}, x_i)$ for which

$$G(x_i) - G(x_{i-1}) = G'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1}).$$

Hence, the Riemann sum R_n is

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (G(x_i) - G(x_{i-1})). \\ &= (G(x_1) - G(x_0)) + (G(x_2) - G(x_1)) + \cdots + (G(x_n) - G(x_{n-1})). \end{aligned}$$

This sum telescopes to

$$R_n = G(x_n) - G(x_0) = G(b) - G(a).$$

Because f is continuous, we know that $\int_a^b f(x)\,dx$ exists and all Riemann sums approach this number. We also know that, for the particular choices of evaluation points made with the help of the Mean-value Theorem, all Riemann sums are equal to the number $G(b) - G(a)$. Hence,

$$\int_a^b f(x)\,dx = \lim_{x\to\infty} R_n = \lim_{n\to\infty}(G(b) - G(a)) = G(b) - G(a).$$

This is (10), which we were to prove.

We review the Fundamental Theorem in the following examples.

EXAMPLE 1 Find the area of the region beneath the first arch of the sine curve.

Solution

Figure 5.20 shows the region $\mathcal{R}$ whose area we wish to calculate. We calculated the area of this region in Example 4 of Section 5.2. The calculation that follows is *much* easier! The area of $\mathcal{R}$ is the value of the integral $\int_0^{\pi} \sin x\, dx$. From (10), the value of the integral is $G(\pi) - G(0)$, where G is an antiderivative of $\sin x$. Because $d(-\cos x)/dx = \sin x$, we may take $G(x) = -\cos x$. From (10),

$$\int_0^{\pi} \sin x\, dx = G(\pi) - G(0) = -\cos \pi - (-\cos 0) = -(-1) - (-1) = 2.$$

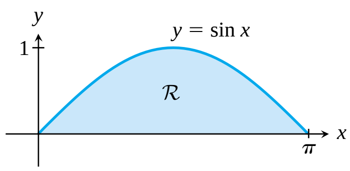

FIGURE 5.20 The region beneath the first arch of the sine curve.

The preceding calculation depended on both the idea of the integral and the Fundamental Theorem of Calculus. The first step was our recognition that the integral can be used to calculate an area. The second step was the use of the Fundamental Theorem to evaluate the integral by determining an antiderivative. The integral and the Fundamental Theorem, working together, provide us with a powerful tool for calculating the areas of planar regions, the volumes of solids of revolution, and the length of curves.

Java Applet
Definite Integrals
Allows students to explore the connection between antiderivatives and areas under a curve, including positive and negative areas. Graphs the area under a function over an interval and computes the area using the given antiderivative of the function.

WEB

In the next example we use the convenient "evaluation notation" $G(x)\big|_a^b$, which, if G is a function defined on an interval $[a, b]$, denotes the number $G(b) - G(a)$, that is,

$$G(x)\Big|_a^b = G(b) - G(a).$$

EXAMPLE 2 Find the area of the region $\mathcal{R}$ beneath the graph of

$$f(x) = (x-1)^2, \qquad 0 \le x \le 3.$$

Solution

A good first step is to sketch the region whose area we wish to calculate. Investing time and effort here often clarifies the problem, exposes subproblems whose solutions we must find, and helps avoid major blunders. Figure 5.21 shows the graph of f. We added the line with equation $x = 3$ and shaded the region $\mathcal{R}$. The area of $\mathcal{R}$ is the value of the integral

$$\int_0^3 (x-1)^2\, dx.$$

An antiderivative of the integrand is $\frac{1}{3}(x-1)^3$. Hence,

$$\int_0^3 (x-1)^2\, dx = \frac{1}{3}(x-1)^3\Big|_0^3 = \frac{1}{3}(3-1)^3 - \frac{1}{3}(0-1)^3 = \frac{8}{3} + \frac{1}{3} = 3.$$

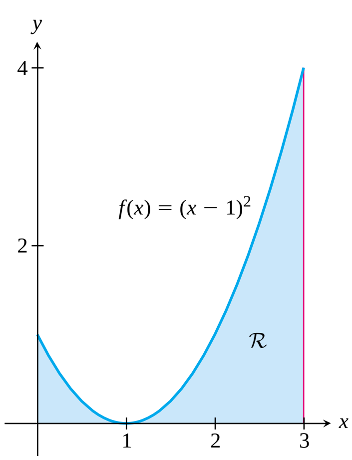

FIGURE 5.21 The region beneath the graph of $f(x) = (x-1)^2$, $0 \le x \le 3$.

Part II of the Fundamental Theorem of Calculus was used in these two examples. Next we give two examples of Part I.

EXAMPLE 3 Let A be the area function defined by

$$A(x) = \int_0^x \sin w\,dw, \qquad 0 \le x \le \pi. \tag{14}$$

Determine $A'(\pi/4)$.

Solution

We calculated the area of the region beneath the first arch of the sine function in Example 1. Here we are asked to consider the area function A defined in (14) and illustrated in Fig. 5.22. The value of $A(x)$ for a representative x is the area of the region beneath the sine curve and between the lines $w = 0$ and $w = x$. In Example 1 we calculated $A(\pi) = 2$ by noting that $-\cos w$ is an antiderivative of $\sin w$. From Part II of the Fundamental Theorem of Calculus,

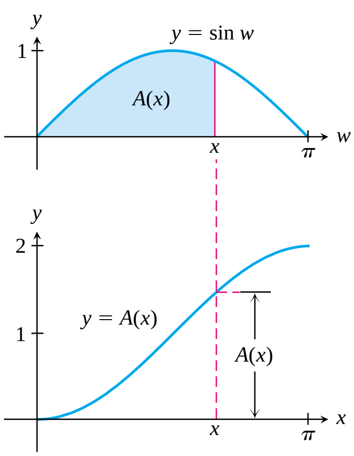

FIGURE 5.22 The region beneath the curve $y = \sin w$, from $w = 0$ to $w = x$.

$$A(x) = \int_0^x \sin w\,dw = -\cos w\Big|_0^x = 1 - \cos x. \tag{15}$$

Hence,

$$\begin{aligned} A'(x) &= 0 - (-\sin x) = \sin x \\ A'\left(\frac{\pi}{4}\right) &= \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}. \end{aligned} \tag{16}$$

This calculation can be done more easily by using Part I of the Fundamental Theorem of Calculus. From (9),

$$A'(x) = \frac{dA}{dx} = \frac{d}{dx}\left(\int_0^x \sin w\,dw\right) = \sin x. \tag{17}$$

This agrees with (16) and avoids finding an antiderivative needed for (15).

In the next example we show how a rate of change can be found using the chain rule together with Part I of the Fundamental Theorem of Calculus.

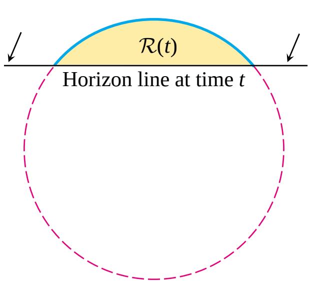

FIGURE 5.23 The region $\mathcal{R}(t)$ of the rising moon at time t after moonrise.

EXAMPLE 4 Figure 5.23 shows a schematic of the moon rising over the eastern horizon. At time t after the start of moonrise, the area $A(t)$ of the region $\mathcal{R}(t)$ of the moon above the horizon is

$$A(t) = \pi a^2 - 2\int_{-a}^{a-kt} \sqrt{a^2 - w^2}\,dw, \qquad 0 \le t \le \frac{2a}{k}, \tag{18}$$

where a is the radius of the moon and k is a positive constant. Find the rate of change $A'(t)$ of $A(t)$ at time $t = a/(2k)$.

Solution

Although it is possible to find an antiderivative for $\sqrt{a^2 - w^2}$ and obtain an explicit expression for $A(t)$ by applying Part II of the Fundamental Theorem, it is far easier to apply Part I. For this we rewrite (18) as

$$A(t) = \pi a^2 - 2\int_{-a}^{x} \sqrt{a^2 - w^2}\, dw, \quad \text{where } x = a - kt. \tag{19}$$

By the chain rule,

$$A'(t) = \frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt}. \tag{20}$$

Because $x = a - kt$, $dx/dt = -k$. Calculating the derivative dA/dx by Part I of the Fundamental Theorem, we may write (20) as

$$\begin{aligned} A'(t) &= -k \cdot \frac{d}{dx}\left(\pi a^2 - 2\int_{-a}^{x} \sqrt{a^2 - w^2}\, dw\right) \\ &= -k \cdot \left(0 - 2\sqrt{a^2 - x^2}\right) = 2k\sqrt{a^2 - x^2}. \end{aligned}$$

Replacing x by $a - kt$,

$$A'(t) = 2k\sqrt{a^2 - (a - kt)^2}.$$

At $t = a/(2k)$

$$A'\left(\frac{a}{2k}\right) = 2k\frac{a\sqrt{3}}{2} = ka\sqrt{3} \text{ square units per time unit.}$$

We end the section by listing in the summary that follows several other properties and definitions relating to the integral. For completeness, we have included earlier properties and definitions as well. In Definition 1 we omit the definition of *net area.*

Definitions and Properties of the Integral

Assume that f and g are continuous on $[a, b]$, c is a real number, and r, s, $t \in [a, b]$.

Definition 1: Area. Let f be a nonnegative function on $[a, b]$. The area of the region $\mathcal{R}$ beneath the graph of f is the number $\int_a^b f(x)\, dx$.

Definition 2: $\displaystyle\int_a^a f(w)\, dw = 0.$

Definition 3: $\displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx.$

Definition 4: Average value. The average value of f on $[a, b]$ is

$$\frac{1}{b - a}\int_a^b f(x)\, dx.$$

Property 1: Additivity.

$$\int_r^s f(x)\,dx + \int_s^t f(x)\,dx = \int_r^t f(x)\,dx.$$

Property 2: Mean-value Theorem for Integrals. There is a point $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b - a).$$

Property 3: Linearity.

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Property 4: $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$

COMMENTS ON DEFINITION 3 This definition, together with Definition 2, makes it possible to state Property 1 for any choices of r, s, $t \in [a, b]$, not just for $a \leq r < s < t \leq b$.

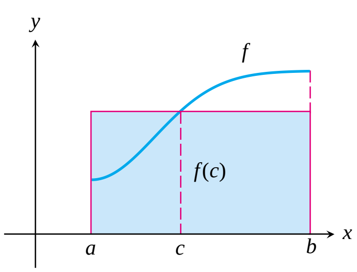

FIGURE 5.24 Mean-value Theorem for

COMMENTS ON DEFINITION 4 Note from Property 2 that the average value of f is $f(c)$. Figure 5.24, taken from our discussion of the Mean-value Theorem for Integrals, shows that the function value $f(c)$ is an average value of f in the sense that the rectangle with width $b - a$ and height $f(c)$ has the same area as the region beneath the graph of f.

COMMENTS ON PROPERTY 3 Functions like $L(x) = mx$ are called *linear* because

$$L(x_1 + x_2) = m(x_1 + x_2) = mx_1 + mx_2 = L(x_1) + L(x_2) \tag{21}$$

and

$$L(bx) = m(bx) = b(mx) = bL(x). \tag{22}$$

The derivative is called *linear* because the derivative of a sum of two functions is the sum of their derivatives and the derivative of a scalar times a function is the product of the scalar and the derivative of that function. These properties are analogous to properties (21) and (22).

The definite integral is called *linear* because it satisfies properties analogous to properties (21) and (22). The two parts of Property 3 may be stated in words: The integral of a sum is the sum of the integrals, and the integral of a scalar times a function is the scalar times the integral of the function.

We show in Exercises 49 and 50 that the integral satisfies Property 3.

PROPERTY 4 We show in Exercises 51 and 52 that the integral satisfies Property 4.

Exercises 5.3

Exercises 1–12: Sketch the graph of the function and use Part II of the Fundamental Theorem of Calculus in calculating the area of the region beneath the graph and between the lines described by $x = a$ and $x = b$.

1. $f(x) = \cos x,\ a = 0,\ b = \pi/2$

2. $f(x) = \sin x,\ a = \pi/6,\ b = \pi/2$

3. $f(x) = (x - 1)^3,\ a = 1,\ b = 3$

4. $f(x) = \frac{1}{3}(x - 3)^4,\ a = 2,\ b = 5$

5. $f(x) = e^x,\ a = 0,\ b = 1$

6. $f(x) = 2^x,\ a = 0,\ b = 1$

7. $f(x) = \sec^2 x,\ a = 0,\ b = \pi/4$

8. $f(x) = \sec x \tan x,\ a = 0,\ b = \pi/3$

9. $f(x) = 1/(x^2 + 1),\ a = 0,\ b = \pi/4$

10. $f(x) = 1/\sqrt{1 - x^2},\ a = 0,\ b = 1/2$

11. $f(x) = 1/x,\ a = 1,\ b = 2$

12. $f(x) = 1/\sqrt{x},\ a = 1,\ b = 4$

Exercises 13–22: Use Part I of the Fundamental Theorem of Calculus in differentiating the integral. Use the chain rule as in Example 4 when the upper limit is not simply x.

13. $\int_1^x \ln w\, dw$

14. $\int_0^x \sin w^2\, dw$

15. $\int_1^x \sqrt{1 + t^2}\, dt$

16. $\int_0^x \sqrt{1 - t^2}\, dt$

17. $\int_1^{x^2} \ln w\, dw$

18. $\int_0^{x^3} \cos\sqrt{w}\, dw$

19. $\int_1^{\sqrt{x}} \sqrt{1 + t^2}\, dt$

20. $\int_0^{1/x} \tan t^2\, dt$

21. $\int_0^{e^x} (\arctan t - t^2)\, dt$

22. $\int_0^{\sin x^2} \sin t^2\, dt$

Exercises 23–34: Find an antiderivative of the function f defined on $a \le x \le b$ and then evaluate $\int_a^b f(x)\, dx$.

23. $f(x) = \sqrt{x},\ 0 \le x \le 4$

24. $f(x) = x^{4/3},\ 0 \le x \le 8$

25. $f(x) = x^3 - 7x,\ 0 \le x \le 1$

26. $f(x) = x - 2x^4,\ 0 \le x \le 1$

27. $f(x) = x(3x - 1),\ 0 \le x \le 4$

28. $f(x) = (x + 1)(x^2 + 1),\ 0 \le x \le 2$

29. $f(x) = (x^2 + 1)/x,\ 1 \le x \le 3$

30. $f(x) = (x^3 + 2x + 1)/x^2,\ 1 \le x \le 4$

31. $f(x) = x^{1/3} - 2x^{-2/3},\ 1 \le x \le 8$

32. $f(x) = x^{-1/4} - 7\sqrt{x},\ 1 \le x \le 16$

33. $f(x) = 2x \sin x^2,\ 0 \le x \le 1$

34. $f(x) = 2xe^{x^2},\ 0 \le x \le 1$

Exercises 35–38: For the given function f, let $F(x) = \int_a^x f(w)\, dw$. Evaluate F at the given points.

35. f as defined in the accompanying graph; $a = 2$; $F(2)$, $F(3)$, and $F(5)$

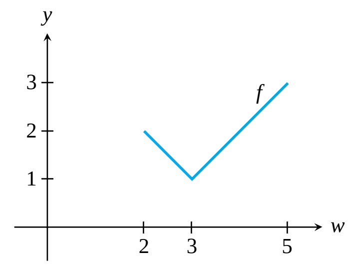

Figure for Exercise 35.

36. f as defined in the accompanying graph; $a = 0$; $F(0)$, $F(1)$, and $F(2)$

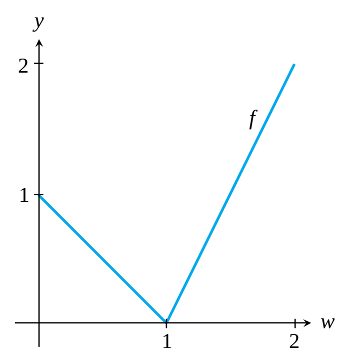

Figure for Exercise 36.

37. $f(w) = \sin w,\ 0 \le w \le \pi$; $a = 0$; $F(0)$, $F(\pi/4)$, $F(\pi/2)$, and $F(\pi)$

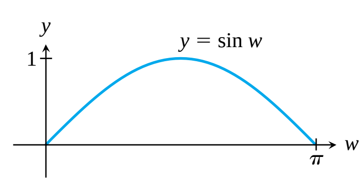

Figure for Exercise 37.

38. $f(w) = \cos w$, $0 \le w \le \pi$; $a = 0$; $F(0)$, $F(\pi/2)$, and $F(\pi)$

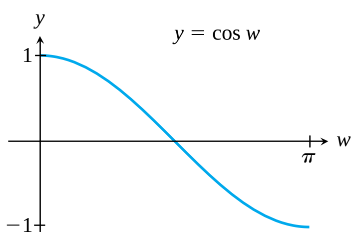

Figure for Exercise 38.

Exercises 39–44: A function f has an antiderivative F. Calculate $\int_a^b f(x)\,dx$.

39. $F(x) = x + 1/x$; $a = 1$, $b = 3$

40. $F(x) = 1/\sqrt{x^2 + 1}$; $a = 0$, $b = \sqrt{8}$

41. $F(x) = \sin(x^2)$; $a = 0$, $b = 1$

42. $F(x) = \tan(\sqrt{x})$; $a = 0$, $b = 4$

43. $F(x) = e^{x^2}$; $a = 0$, $b = 1$

44. $F(x) = 2^x$; $a = 1$, $b = 4$

45. What is the average value of the cosine function on $[0, \pi/2]$? Determine the height of a rectangle with base the interval $[0, \pi/2]$ and area equal to that under the graph of the cosine function on $[0, \pi/2]$. Sketch the graph of the cosine function and the rectangle. Write a simple sentence or two about the meaning of the sketch.

46. What is the average value of the function $f(x) = e^x$ on $[0, 2]$? Determine the height of a rectangle with base the interval $[0, 2]$ and area equal to that under the graph of the exponential function on $[0, 2]$. Sketch the graph of f on $[0, 2]$ and the rectangle. Write a simple sentence or two about the meaning of the sketch.

47. Show that (12) follows from Property 1. *Hint:* Apply Property 1 to justify

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(w)\,dw - \int_a^x f(w)\,dw \\ &= \int_x^a f(w)\,dw + \int_a^{x+h} f(w)\,dw \\ &= \int_x^{x+h} f(w)\,dw. \end{aligned}$$

48. Complete the proof of Part I of the Fundamental Theorem of Calculus by working through the case in which $h < 0$. *Hint:* See the preceding exercise.

49. Use the Fundamental Theorem of Calculus in verifying the first part of Property 3. *Hint:* Let F and G be antiderivatives of f and g, and write an antiderivative of $f + g$ in terms of F and G.

50. Use the Fundamental Theorem of Calculus in verifying the second part of Property 3. *Hint:* Let F be an antiderivative of f, express $c\int_a^b f(x)\,dx$ in terms of F, and, finally, express an antiderivative of cf in terms of F.

51. Make a convincing argument that if f is continuous and nonnegative on $[a, b]$, then $\int_a^b f(x)\,dx \ge 0$.

52. Apply the result stated in the preceding exercise to prove Property 4 of the integral. You may use the following outline: First, observe that $|f(x)| \ge f(x)$ on $[a, b]$; infer that $|f(x)| - f(x) \ge 0$ on $[a, b]$; apply Exercise 51 to show that $\int_a^b |f(x)|\,dx \ge \int_a^b f(x)\,dx$. Next, observe that $|f(x)| \ge -f(x)$ on $[a, b]$; infer that $|f(x)| + f(x) \ge 0$ on $[a, b]$; apply Exercise 51 to show that $\int_a^b |f(x)|\,dx \ge -\int_a^b f(x)\,dx$. Finally, infer Property 4.

53. The statements of Property 1 near Equation (5) and in the summary of the definitions and properties of the integral are not the same. Show that the more general statement follows from Equation (5).

54. Fill in the details of the following proof that for all $a, b > 0$

$$\lim_{x\to\infty} \frac{(\ln x)^a}{x^b} = 0.$$

This limit was given in Section 4.4. Let $p > 0$; then for $w \ge 1$, $w^{-1} \le w^{-1+p}$. By the result stated in Exercise 51,

$$\int_1^x \frac{1}{w}\,dw \le \int_1^x w^{-1+p}\,dw.$$

Hence

$$\ln x \le \frac{x^p}{p} - \frac{1}{p} < \frac{x^p}{p}.$$

Using this result, we have

$$\frac{(\ln x)^a}{x^b} < \frac{x^{pa}}{p^a x^b}.$$

After showing that p can be chosen so that $pa - b < 0$, let $x \longrightarrow \infty$ and infer the desired result.

55. Let F be defined on $[0, 1.5]$ by

$$F(x) = \int_0^{x^2} \frac{\sin t}{t+1}\,dt.$$

Where does the maximum value of F occur?

T **56.** Use your calculator or CAS to generate 100 random numbers in the interval $[0, \pi]$. Calculate the sines of these numbers and average the 100 function values thus generated. This number approximates the average value of the sine function on $[0, \pi]$. Most calculators have a built-in random number generator that returns numbers randomly chosen from $[0, 1]$. A labor-intensive procedure is to press the RAND button and then the SIN button, and add the result to a variable whose initial value is 0. Repeat this 100 times.

Divide the sum by 100. If you can program your calculator, this procedure can be automated. If you repeat the program, say, 20 times and average the outcomes, you should obtain $2/\pi = 0.6366\ldots$, more or less. It may be necessary to arrange for a different "seed" for the random number generator as you repeat the program.

57. The accompanying figure shows a valve opening in a circular pipe of radius 1 meter. At any time $t \in [0, 1]$, the left edge of the valve lies along the line with equation $x = 2t^2$ and water is flowing in the shaded region to the left of this line. At time t, the flow $W(t)$ through the valve, measured in cubic meters per minute, is three times the area of the shaded region. What is the rate of change of the flow at $t = 0.5$ minutes?

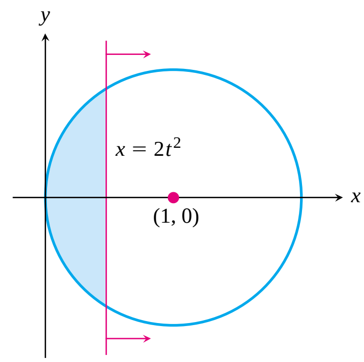

Figure for Exercise 57.

5.4 The Indefinite Integral

To evaluate a definite integral $\int_a^b f(x)\,dx$ it is sufficient to find an antiderivative F of f because, by the Fundamental Theorem of Calculus,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

For this reason our main efforts in Sections 5.5–5.8 will be to explore ways of finding an antiderivative F for a given function f. To facilitate this discussion, it is convenient to introduce the idea of the *indefinite integral*.

A function F is an antiderivative of f provided that the derivative of F is f; if we have a variable like x in mind, we may write $F'(x) = f(x)$. This language is descriptive but awkward. In what follows, we use the notation

$$\int f(x)\,dx$$

for an antiderivative of $f(x)$. This notation frees us from choosing a letter like F to denote an antiderivative of f. Though usually not made explicit, when we use this notation we understand that there is an interval I on which f is defined and continuous.

While the definite integral $\int_a^b f(x)\,dx$ is a number, the indefinite integral $\int f(x)\,dx$ denotes a function, one whose derivative is $f(x)$ for all x in some interval I. The term "integral" may refer to either an indefinite or a definite integral.

Here are some examples:

$$\int x^2\,dx = \tfrac{1}{3}x^3,$$

$$\int \cos x\,dx = \sin x,$$

$$\int \frac{1}{x}\,dx = \ln x,$$

and, more generally,

$$\int f(x)\,dx = \int_a^x f(w)\,dw.$$