"Expanding" this sum by using Properties (1) and (2),

$$1^3 - (m+1)^3 = -3\sum_{i=1}^{m} i^2 - 3\sum_{i=1}^{m} i - \sum_{i=1}^{m} 1.$$

Solving this expression for $\sum_{i=1}^{m} i^2$,

$$\sum_{i=1}^{m} i^2 = \frac{1}{3}\left((m+1)^3 - 1 - 3\sum_{i=1}^{m} i - \sum_{i=1}^{m} 1\right).$$

Expanding $(m+1)^3$ and using (4) and (5), which we proved in the first two examples,

$$\sum_{i=1}^{m} i^2 = \frac{1}{3}\left(m^3 + 3m^2 + 3m - \frac{3}{2}m(m+1) - m\right)$$
$$= \frac{1}{6}(2m^3 + 3m^2 + m).$$

Hence,

$$\sum_{i=1}^{m} i^2 = \frac{m(2m^2+3m+1)}{6} = \frac{m(m+1)(2m+1)}{6}.$$

This is (6).

Exercises 5.1

Exercises 1–14: Rewrite using summation notation. Factor out any common factors.

1. $1 + 1/2 + 1/3 + \cdots + 1/10$
2. $1 + 1/2^1 + 1/2^2 + \cdots + 1/2^{10}$
3. $-\ln(1/1) - \ln(1/2) - \cdots - \ln(1/100)$
4. $-e^{1+1/1} - e^{1+1/2} - \cdots - e^{1+1/10000}$
5. $1^2 + 2^2 + \cdots + 10^2$
6. $1^{1/1} + 2^{1/2} + \cdots + 7^{1/7}$
7. $2/3 + 2/4 + 2/5 + \cdots + 2/35$
8. $7/13 + 8/13 + 9/13 + \cdots + 27/13$
9. $\sqrt{h^2(1/19)} + \sqrt{h^2(2/19)} + \cdots + \sqrt{h^2(21/19)}$
10. $\sqrt{3/h^2} + \sqrt{4/h^2} + \cdots + \sqrt{37/h^2}$
11. $\pi^{1+1} + \pi^{1+2} + \pi^{1+3} + \cdots + \pi^{1+21}$
12. $0.01 \cdot \ln(1) + 0.01 \cdot \ln(2) + \cdots + 0.01 \cdot \ln(50)$
13. $1/2 + 2/3 + 3/4 + \cdots + 15/16$
14. $x^1 + x^2 + \cdots + x^n$

Exercises 15–26: Expand each sum and evaluate. For example, $\sum_{i=1}^{4} i^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100.$

15. $\sum_{i=1}^{5} i^2$
16. $\sum_{i=1}^{3} i^4$
17. $\sum_{i=1}^{6} (i^2 + 1)$
18. $\sum_{i=1}^{3} (2i^2 - 5i + 1)$
19. $\sum_{j=1}^{2} \sqrt{j+1}$
20. $\sum_{j=1}^{4} \sqrt{3j-2}$
21. $\sum_{k=0}^{5} \sin(k/10)$
22. $\sum_{k=0}^{5} \cos(k/10)$

23. $\sum_{i=0}^{5} 1/(2i+1)$ **24.** $\sum_{i=0}^{5} (3i-5)/3$

25. $\sum_{j=1}^{3} j^0$ **26.** $\sum_{j=1}^{3} 1$

Exercises 27–40: Evaluate each sum. Use (4)–(8) as needed.

27. $\sum_{j=1}^{25} j^2$ **28.** $\sum_{j=1}^{25} 3j^3$

29. $\sum_{j=1}^{10} j(j+1)$ **30.** $\sum_{j=1}^{10} 2j(j-1)$

31. $\sum_{j=1}^{m} j(2j+3)$ **32.** $\sum_{j=1}^{m} j(3j-1)$

33. $\sum_{j=1}^{m} 5j(3-j^2)$ **34.** $\sum_{j=1}^{m} 3j(2-j^2)$

35. $\sum_{j=1}^{m} j^2+j+1$ **36.** $\sum_{j=1}^{m} j^2-j+1$

37. $\sum_{j=1}^{m} j(7j+3)^2$ **38.** $\sum_{j=1}^{m} j(2j-1)^2$

39. $\sum_{j=3}^{m} (2j+1)^4$ **40.** $\sum_{j=3}^{m} (3j+5)^4$

Exercises 41–44: Use mathematical induction to prove that each formula holds for $m = 1, 2, \ldots$. Proving a statement S_m by mathematical induction has two parts: (1) Show that S_1 is true; (2) show that, for any $k \geq 1$, if statement S_k is true, then so is S_{k+1}. Note that (1) and (2) together show that S_2 is true; applying (2) again shows that S_3 is true; and so on. A brief discussion of mathematical induction is given in the Appendix.

41. $\sum_{i=1}^{m} i = \dfrac{m(m+1)}{2}$

42. $\sum_{i=1}^{m} i^2 = \dfrac{m(m+1)(2m+1)}{6}$

43. $\sum_{i=1}^{m} i^3 = \dfrac{m^2(m+1)^2}{4}$

44. $\sum_{i=1}^{m} i^4 = \dfrac{m(m+1)(2m+1)(3m^2+3m-1)}{30}$

45. Use the distributive property $a(b+c) = ab+ac$ in showing that (2) holds.

46. Prove that (3) holds using the following outline: First show that $|a+b| \leq |a| + |b|$ by considering cases $a, b \geq 0$; $a \geq 0, b < 0$; $a < 0, b \geq 0$; and $a < 0, b < 0$. Next, show that $|a+b+c| \leq |a| + |b| + |c|$, etc. *Hint:* Let $b + c = B$.

47. Show that the identity

$$\sin\alpha \sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

holds by adding together expressions for $\cos(\alpha-\beta)$ and $\cos(\alpha+\beta)$.

48. Use the product-to-sum identity in Exercise 47 in proving (9), which is: for $m = 1, 2, \ldots$ and $1/2x \neq 0, \pm\pi, \pm 2\pi, \ldots$,

$$\sum_{i=1}^{m} \sin(ix) = \frac{\sin\left(\frac{1}{2}mx\right)\sin\left(\frac{1}{2}(m+1)x\right)}{\sin\left(\frac{1}{2}x\right)}.$$

Multiply both sides of this equation by $\sin\left(\frac{1}{2}x\right)$ and apply the product-to-difference identity to show that

$$\sum_{i=1}^{m} \sin(ix)\sin\left(\tfrac{1}{2}x\right) = \frac{1}{2}\sum_{i=1}^{m}\Big(\cos\left(\tfrac{1}{2}(2i-1)x\right) - \cos\left(\tfrac{1}{2}(2i+1)x\right)\Big).$$

Expand this sum and notice that because the terms adjacent to the + signs add to 0, the sum "telescopes" to two terms:

$$\frac{1}{2}\sum_{i=1}^{m}\Big(\cos\left(\tfrac{1}{2}(2i-1)x\right) - \cos\left(\tfrac{1}{2}(2i+1)x\right)\Big) = \frac{1}{2}\Big(\cos\left(\tfrac{1}{2}x\right) - \cos\left(\tfrac{1}{2}(2m+1)x\right)\Big).$$

Apply the product-to-difference identity again.

49. Show that

$$\sum_{i=1}^{m} \cos(ix) = \frac{\sin\left(\frac{1}{2}mx\right)\cos\left(\frac{1}{2}(m+1)x\right)}{\sin\left(\frac{1}{2}x\right)}.$$

Hint: Use the (other) product-to-sum identity

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$$

and adapt the outline in Exercise 48.