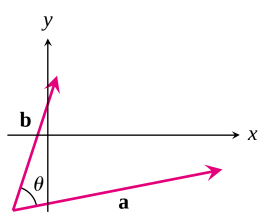
Curve, Tangent Vector, Slope of Tangent Line (*continued*)

From the Tangent Vector and Slope Theorem, the slope of the tangent line L to C at $\mathbf{r}(t)$ is

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

$$\frac{dy}{dx} = \left.\frac{dy/dt}{dx/dt}\right|_{t=3\pi/4} = \frac{3\pi - 4}{3\pi + 4} \approx 0.404.$$

Dot Product and Angle between Vectors



The dot product of vectors $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

The angle θ between $\mathbf{a}$ and $\mathbf{b}$ can be calculated from

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}.$$

Nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

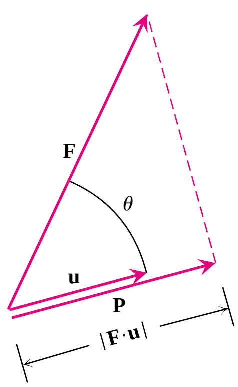
The dot product of the vectors

$$\mathbf{a} = \langle 28.1, 5.4\rangle \quad \text{and} \quad \mathbf{b} = \langle 6.0, 18.2\rangle,$$

is $\mathbf{a} \cdot \mathbf{b} = 266.88$. The angle between them is

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right) \approx 61^\circ.$$

Projection of a Vector on a Unit Vector



The projection $\mathbf{P}$ of a vector $\mathbf{F}$ onto a unit vector $\mathbf{u}$ is

$$\mathbf{P} = (\mathbf{F} \cdot \mathbf{u})\mathbf{u}.$$

The length of this projection is

$$\|\mathbf{P}\| = |\mathbf{F} \cdot \mathbf{u}| = \|\mathbf{F}\|\,|\cos\theta|,$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{u}$.

For $\mathbf{F} = \langle 1, 2\rangle$ and $\mathbf{u} = \langle \cos 15^\circ, \sin 15^\circ\rangle$,

$$\mathbf{P} = (\mathbf{F} \cdot \mathbf{u})\mathbf{u} \approx 1.48\mathbf{u} \approx \langle 1.43, 0.38\rangle.$$

The length of this projection is

$$\|\mathbf{P}\| \approx 1.48.$$

Chapter Review Exercises

1. Give the definitions of average velocity for objects moving along a line and for objects moving in the plane.

2. An object is moving on the x-axis and its coordinate positions at $t = 36$ s and $t = 39$ s are $x = 35.8$ m and $x = 24.7$ m. What is its average velocity during this 3-second interval?

3. An object is moving in the (x, y)-plane and its position vectors at $t = 5.5$ s and $t = 7.2$ s are $\langle -10, 3\rangle$ and $\langle 1, 4\rangle$. What is its average velocity during this 1.7-second interval?

4. The position x of the point of contact between a freight car and a snubber (see Section 3.1, Example 3) is

$$x = x(t) = -0.5e^{-0.25t}\sin(1.5t),$$

for all times between the time $t = 0$ of first contact and the time of last contact. Calculate the velocity of the freight car at $t = 2.0$. What is the time of last contact?

5. A rifle is fired vertically upward so that at $t = 0$ the bullet is 20 m below ground level and has speed 610 m/s. Calculate the position, velocity, and acceleration of the bullet under the assumption of no air resistance. What are the bullet's speeds at 1000 m above ground level on the way up and on the way down?

6. Approximately 2.5 hours ago, your odometer broke. It read 99,999 miles at the time. From a close analysis of available speedometer data, it appears that the true velocity function was very nearly

$$v(t) = -26t^3 + 117t^2 - 156t + 65,$$

for $0 \le t \le 2.5$ h. What would the odometer reading be now had it not broken?

7. An object is moving upward and to the right with speed 10 m/s on the line through $(-20.5, -4.8)$ and $(1.5, 12.3)$. If the object was first noticed 15 s ago as it passed through the point $(-20.5, -4.8)$, where is it now?

8. Two tugboats exert forces

$$\mathbf{F}_1 = 120.5\mathbf{i} + 100.5\mathbf{j} \text{ kN}$$

and

$$\mathbf{F}_2 = 150.3\mathbf{i} - 30.1\mathbf{j} \text{ kN}$$

on a barge. What single force on the tug would just balance the combined force of the two tugs? (A kilonewton (kN) is 1000 newtons.)

9. Calculate the coordinates of the two unit vectors making an angle of 0.5 radians with the vector $\langle 6, 1\rangle$.

10. An object at point P is displaced by the vector $\mathbf{a}_1 = \langle 3, 4\rangle$ and then further displaced by $\langle -10, 7\rangle$, $\langle 2, -8\rangle$, and $\langle 20, 0\rangle$. What is the single equivalent displacement of the object? In what direction and distance from P is the object moved by the four displacements?

11. The hyperbolic functions cosh and sinh are defined by $\cosh(x) = (e^x + e^{-x})/2$ and $\sinh(x) = (e^x - e^{-x})/2$. Let C be the curve described by

$$\mathbf{r} = \mathbf{r}(t) = \langle \cosh t, \sinh t\rangle, \qquad t \ge 0.$$

Sketch C. Note that $\cosh^2 t - \sinh^2 t = 1$ for all t.

12. Sketch the graph of the curve C described by

$$\mathbf{r} = \mathbf{r}(t) = \langle 1 - \cos t, -1 + \sin t\rangle,$$

where $0 \le t \le 2\pi$. Find the coordinates of all points on C at which the slope is $2/3$.

13. Particle A moves on the path described by $\mathbf{r}(s) = \langle 2s, -5 + s\rangle$, $s \ge 0$, while particle B moves on the path described by $\mathbf{r} = \langle t + t^2, t - t^2\rangle$, $t \ge 0$. Do the particles collide if both s and t are read from the same clock?

14. Calculate the angle between the lines with equations

$$\mathbf{r} = \langle 2, 1\rangle + t\langle 3, 4\rangle$$

and

$$\mathbf{r} = \langle -1, 5\rangle + t\langle -3, 1\rangle.$$

15. Show that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and $\mathbf{a} \ne \pm\mathbf{b}$.

16. Find parametric equations describing the two lines tangent to the graph of the equation $y = x^3$ and passing through the point $(1, 1)$. *Hint:* Choose t so that the line described by

$$\mathbf{r}(s) = \langle t, t^3\rangle + s\langle 1, 3t^2\rangle,$$

where $-\infty < s < \infty$, passes through the point $(1, 1)$.

17. The position vector of an object is

$$\mathbf{r}(t) = \langle \cos 2t, \sin 2t\rangle, \qquad t \ge 0.$$

Lengths are in meters and time in seconds. Calculate its acceleration at $t = 3.1$ s.

18. An object's acceleration is $\mathbf{a} = \langle e^t, e^{-t}\rangle$, $t \ge 0$. If $\mathbf{r}(0) = \langle 0, 0\rangle$ and $\mathbf{v}(0) = \langle 1, 0\rangle$, find its position at any time $t \ge 0$.

19. A 1.2-kg object initially at rest at the origin is acted on by a force $\mathbf{F} = \langle 2.4, 1.7\rangle$ N. What is the object's acceleration? Where is the object and how fast is it moving 3.5 s after the force is first applied?

20. The mass of an electron is 9.11×10^{-31} kg. Calculate the gravitational force exerted by one electron on another if they are 1 mm apart.

21. Write a parametric equation for the circle with center at $(-2, 3)$, radius 5, and traversed in the counterclockwise direction.

22. Calculate the slope of the curve described by

$$\mathbf{r}(t) = \langle t^3 + 1, t^2 + t + 1\rangle, \qquad t \ge 0$$

at the point $\mathbf{r}(2)$. Give a parametric equation of the tangent line to C at this point.

23. Let $\mathbf{e}_1 = \langle 3, 1\rangle$, $\mathbf{e}_2 = \langle -1, 3\rangle$, and $\mathbf{v} = \langle 5, 7\rangle$. Express $\mathbf{v}$ as a sum of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel to $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively.

24. For what value of t are the vectors $\mathbf{a} = \langle 3, 1\rangle$ and $\mathbf{b} = \langle -2, t\rangle$ parallel? Perpendicular?

25. Use vectors to show that any angle inscribed in a semicircle is a right angle.

26. Calculate $\|\mathbf{a} + \mathbf{b}\|$ and $\|\mathbf{a} - \mathbf{b}\|$, given that $\|\mathbf{a}\| = 5$, $\|\mathbf{b}\| = 8$, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $2\pi/3$.

T **27.** Sketch the graph of the curve C described by the parametric equation

$$\mathbf{r} = \mathbf{r}(t) = \langle t - t^2, t + 2t^2 \rangle,$$

where $-3 \le t \le 3$. Eliminate the parameter to obtain an equation in x and y. Show that the curve described by this equation includes the curve C. Calculate the slope of C at the point $(-2, 10)$.

28. Let $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$ describe a line L, and let $\mathbf{q}$ be the position vector of a point Q not on L. Show that the distance d from L to Q can be calculated as follows. Let $\mathbf{p}$ be the vector projection of $\mathbf{q} - \mathbf{r}_0$ onto $\mathbf{a}$ and $\mathbf{n} = \mathbf{q} - \mathbf{r}_0 - \mathbf{p}$. Then $d = \|\mathbf{n}\|$. Use this procedure to calculate the distance from the line through $(1, 5)$ and $(7, 2)$ to the point $(-5, -3)$.

STUDENT PROJECT

A. TIMING A RIFLE BULLET

A method for calculating the time taken in seconds for a rifle ball to travel from muzzle to target was published in the July 1893 issue of *Scientific American*. This was reported in the "50 and 100 Years Ago" column of the July 1993 issue.

> It may be of interest to amateur riflemen to know the following simple method for ascertaining the effect of gravity upon a bullet: Sight the rifle upon the target, keeping the sights plumb above the center line of the bore of the rifle. Mark where the ball strikes. Then reverse the rifle, so as to have the sights exactly beneath the line of the bore. In this reversed position sight it on the target as before, and mark where the bullet strikes. Divide the difference in elevation of the two bullet marks by 32 and extract the square root. This will give the time in seconds that it took the ball to travel the distance. The distance divided by the time will give the speed of the bullet per second.
> –J. A. G., Grand Rapids, Michigan

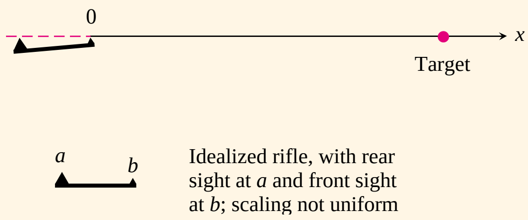


FIGURE 3.62 Diagram for Problem 1.

PROBLEM 1 Verify J. A. G.'s assertions and conclusions, and comment on the assumptions that you and J. A. G. made. See Fig. 3.62.

PROBLEM 2 The word *extract* used by J. A. G. suggests that it was somewhat more difficult to calculate a square root in 1893 than now. Write a short paragraph discussing and contrasting the techniques used by typical students in 1900 and 2000 to calculate, say, $\sqrt{5.73}$.

B. THE QUARTERBACK'S PROBLEM

Figure 3.63 shows a quarterback and receiver at points QB and R on a level playing field. The point R is 15 yards downfield from QB and 10 yards to one side. According to plan A, the receiver will run along the line L at 6 yards per second and receive a pass from the quarterback. The quarterback must pass within 5 seconds after the receiver starts running. The dotted line in the figure is the projection onto the field of the trajectory of the ball in a successful pass. A pass is "successful" if the receiver