is as a function of time t, we can eliminate the parameter t in the parametric equation (2). Letting the coordinates of $\mathbf{r}(t)$ in (2) be

$$x = 30 - 40t$$
$$y = 40 - 30t,$$

we first solve either of these equations for t. From the first equation,

$$t = \frac{30 - x}{40}.$$

Next, we substitute this result into the second equation. This gives

$$y = 40 - 30\left(\frac{30 - x}{40}\right) = \frac{3}{4}x + \frac{70}{4}. \tag{3}$$

Alternatively, recalling that we were given two points $(x_1, y_1) = (30, 40)$ and $(x_2, y_2) = (10, 25)$ on the path of the meteorite, we may use the two-point form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

of a line to obtain equation (3).

Describing Motion on a Circle

INVESTIGATION

Describing the Motion of a Slider-Crank Mechanism

Figure 3.29(a) shows the slider-crank mechanism with which we began this chapter. The circle traced by the point P at the end of the crank (length 5 cm) can be described as the graph of the equation $x^2 + y^2 = 5^2$. This describes the circular path well but does not tell where the pin P is at a given time.

Given that at $t = 0$ the pin was at the point $(5, 0)$ and is moving counterclockwise on the circle at 1 radian per second, it follows that after t seconds the angular displacement of the crank will be $\theta(t) = t$ radians. From (7) in Section 3.2, a position vector of the pin P is

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle 5\cos t, 5\sin t\rangle = 5\langle \cos t, \sin t\rangle. \tag{4}$$

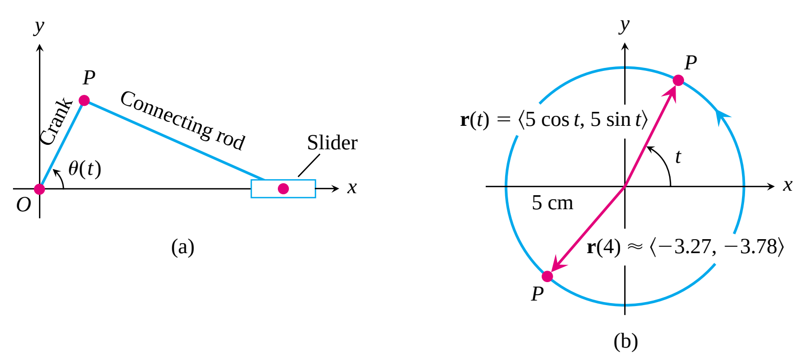

FIGURE 3.29 The location of the pin at P as a function of time t.

From (4), the position of the pin P at any time t can be easily calculated. For example, at $t = 4$ seconds the position of the pin is

$$\mathbf{r}(4) = 5\langle \cos 4, \sin 4 \rangle \approx \langle -3.27, -3.78 \rangle.$$

Figure 3.29(b) shows the position vector $\mathbf{r}(4)$.

Eliminating the Parameter The equation $x^2 + y^2 = 5^2$ describing the circle on which the pin P moves can be recovered from the parametric equation (4) by eliminating the parameter t. Letting $x = 5 \cos t$ and $y = 5 \sin t$, note that

$$x^2 + y^2 = 5^2 \cos^2 t + 5^2 \sin^2 t = 5^2,$$

for all t.

EXAMPLE 3 As of July 1997, the world's tallest Ferris wheel was in Osaka, Japan. See Fig. 3.30. The distance from ground level to its highest point is 112.5 meters and the wheel itself has a diameter of 100 meters. One revolution takes about 15 minutes. Describe the motion of the wheel with a parametric equation and find the maximum speed at which a passenger moves directly downward.

Solution

The center of the ferris wheel will be 50 meters below the high point; hence, a position vector of the center will be $\mathbf{h} = \langle 0, 62.5 \rangle$. We "track" a passenger P on the wheel who, at time $t = 0$, was directly to the right of the center point. Temporarily regarding the center of the ferris wheel as the origin and recalling (7) from Section 3.2, the position vector of P after t minutes will be

$$\mathbf{w} = 50\left\langle \cos\left(\frac{2\pi t}{15}\right), \sin\left(\frac{2\pi t}{15}\right)\right\rangle.$$

The factor $2\pi/15$ comes from the fact that the wheel revolves 2π radians in 15 minutes. We have assumed counterclockwise rotation.

Figure 3.30 shows that the position vector $\mathbf{r} = \mathbf{r}(t)$ of P at any time t is the sum of the constant vector $\mathbf{h}$ and the vector $\mathbf{w}$ with constant length but variable direction, that is,

$$\begin{aligned} \mathbf{r} = \mathbf{r}(t) = \mathbf{h} + \mathbf{w} &= \langle 0, 62.5 \rangle + 50\left\langle \cos\left(\frac{2\pi t}{15}\right), \sin\left(\frac{2\pi t}{15}\right)\right\rangle \\ &= \left\langle 50 \cos\left(\frac{2\pi t}{15}\right), 62.5 + 50 \sin\left(\frac{2\pi t}{15}\right)\right\rangle. \end{aligned} \tag{5}$$

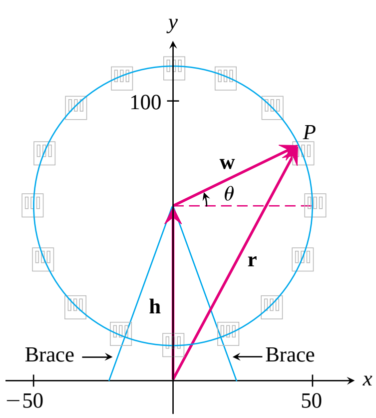

FIGURE 3.30 The position vector **r** of a point on the Ferris wheel, expressed as the sum of a constant vector **h** and a vector **w** that varies with θ.

We can find the vertical speed of the point P by differentiating the y-coordinate of $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t)\rangle$. From

$$y(t) = 62.5 + 50\sin\left(\frac{2\pi t}{15}\right),$$

$$y'(t) = 50\left(\frac{2\pi}{15}\right)\cos\left(\frac{2\pi t}{15}\right) = \left(\frac{20\pi}{3}\right)\cos\left(\frac{2\pi t}{15}\right).$$

Because for all t, $\cos(2\pi t/15)$ is between -1 and 1, the largest value of the speed $|y'(t)|$ is $20\pi/3$ meters per second. This happens at $t = 0$, when cosine is 1 and P is moving upward, and next at $(2\pi/15)t = \pi$, when cosine is -1 and P is moving downward. Hence, the maximum speed directly downward is $20\pi/3 \approx 21$ meters per minute.

Exercises 3.3

Exercises 1–6: A rocket sled is moving in a flat desert. We suppose that a coordinate system has been superimposed on the desert, relative to which the coordinates of the sled at $t = 0$ are (x_0, y_0). We assume that the sled was headed in the given direction at the given speed. Find its position vector $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ at any time t. Calculate its position at time t_1. Sketch the path for $0 \le t \le t_1$. Lengths are in kilometers and time is in hours.

1. $(x_0, y_0) = (0, 1)$; 60° east of north; speed 200 kilometers per hour; $t_1 = 0.001$

2. $(x_0, y_0) = (-1, 0)$; 45° east of north; speed 180 kilometers per hour; $t_1 = 0.002$

3. $(x_0, y_0) = (-5, 3)$; 20° west of north; speed 225 kilometers per hour; $t_1 = 0.001$

4. $(x_0, y_0) = (5, -4)$; 10° north of west; speed 210 kilometers per hour; $t_1 = 0.003$

5. $(x_0, y_0) = (-4.5, 3.2)$; 10° south of east; speed 190 kilometers per hour; $t_1 = 0.001$

6. $(x_0, y_0) = (0.5, -4.6)$; 10° west of south; speed 200 kilometers per hour; $t_1 = 0.002$

Exercises 7–12: A meteorite enters the Earth's atmosphere, moving on a straight line lying in a vertical plane, with the x-axis as the horizon. Its position was recorded at two times. Find its speed and its position vector $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ at any time t before it hits the Earth. Determine where and when it hits Earth. Lengths are in kilometers and time is in seconds.

7. $(50, 40)$ first sighting; $\left(50 - 20\sqrt{3}, 20\right)$ second sighting, 5 seconds later

8. $(60, 30)$ first sighting; $\left(48, 30 - 12\sqrt{3}\right)$ second sighting, 3 seconds later

9. $(35, 30)$ first sighting; $\left(35 - 16\sqrt{2}, 30 - 16\sqrt{2}\right)$ second sighting, 4 seconds later

10. $(-40, 55)$ first sighting; $\left(-40 + 24\sqrt{3}, 31\right)$ second sighting, 6 seconds later

11. $(90.2, 52.7)$ first sighting; $(49.5, 3.3)$ second sighting, 8 seconds later

12. $(-9.9, 65.1)$ first sighting; $(15.9, 34.5)$ second sighting, 5 seconds later

Exercises 13–16: An object is moving on a path described by the parametric equation. Eliminate t to find an equation in x and y describing the path along which the object is moving. Then sketch the path and draw the position vector $\mathbf{r}(t_1)$.

13. $\mathbf{r}(t) = \langle 5t, 2 + 3t\rangle$, $0 \le t \le 1$, $t_1 = 1/2$

14. $\mathbf{r}(t) = (4t - 1)\mathbf{i} - (1 + 3t)\mathbf{j}$, $0 \le t \le 2$, $t_1 = 1$

15. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$, $0 \le t \le 2\pi$, $t_1 = \pi/2$

16. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, $t_1 = 3\pi/2$

Exercises 17–22: Find the position $\mathbf{r}(t)$ of the object at any time t for the described circular path. Sketch.

17. An object P moves counterclockwise from $(5, 3)$ at 2 radians per second on a circle of radius 2 and center $(3, 3)$.

18. An object P moves counterclockwise from $(14, 5)$ at 3 radians per second in a circle of radius 4 and center $(10, 5)$.