3.2 Vectors

In Section 3.1 we studied the motion of objects along a line. In this section and the next we prepare to study motion in a plane or in space. Motion in more than one dimension is most easily described and studied using vectors.

While physical or geometric quantities like mass, speed, and length can be described with a single, positive number, quantities like force, velocity, or displacement, which have both magnitude and direction, cannot be so described. To model such quantities we use vectors, which are often represented graphically by arrows. We start with this idea.

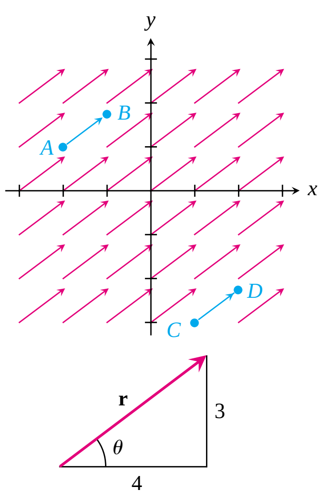


FIGURE 3.11 The flow of a fluid on a plane.

Figure 3.11 is intended to suggest a fluid flowing with constant speed and direction on a horizontal plane during a one-second time interval. The figure both gives a qualitative impression of motion and, if we look more closely, shows a flow in which during the one-second time interval each fluid particle moves the equivalent of four centimeters in the positive x-direction and three centimeters in the positive y-direction. (*Note:* The marks on the axes are four centimeters apart.) The arrows show the *displacements* of several particles of the fluid during the one-second time interval. Because the flow is uniform, all of the arrows have the same length and direction.

We focus on the particle initially at A, with coordinates $(-8, 4)$. After one second, this particle will have been displaced to B, with coordinates $(-8 + 4, 4 + 3) = (-4, 7)$. We describe this displacement with the notation $\overrightarrow{AB}$. Also, shown is the displacement $\overrightarrow{CD}$ of a particle with initial point $(4, -12)$. The terminal point D has coordinates $(4 + 4, -12 + 3) = (8, -9)$. Although $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the displacement of different particles, they are the same in that they have the same length and direction. The distance traveled during one second by each fluid particle is $\sqrt{3^2 + 4^2} = 5$, and each moved in the direction $\theta = \arctan(3/4) \approx 37°$. The arrow or vector $\mathbf{r}$ illustrating this common length and direction is shown in the zoom-view at the bottom of the figure.

Equivalent Vectors; Length and Direction of a Vector

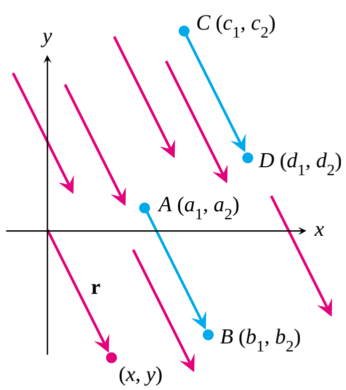


FIGURE 3.12 Equivalent vectors have the same length and the same direction.

Figure 3.12 shows a vector $\overrightarrow{AB}$ from (initial point) A to (terminal point) B and a vector $\overrightarrow{CD}$ from C to D. We say that these vectors are **equivalent** if they have the same length and direction. Using the notation in Fig. 3.12, vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if

$$b_1 - a_1 = d_1 - c_1 \quad \text{and} \quad b_2 - a_2 = d_2 - c_2. \tag{1}$$

If these two conditions hold, it follows that the lengths and directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal.

Figure 3.12 also shows the vector $\mathbf{r}$ with initial point at the origin and terminal point (x, y), where $x = b_1 - a_1$ and $y = b_2 - a_2$. This vector is the simplest of the vectors equivalent to $\overrightarrow{AB}$. The vector $\mathbf{r}$ is determined by the two numbers x and y, which are called **coordinates** of $\mathbf{r}$. We denote $\mathbf{r}$ by

$$\mathbf{r} = \langle x, y \rangle.$$

Because $\mathbf{r} = \langle x, y \rangle$ gives the position of the point (x, y), it is often called a **position vector.**

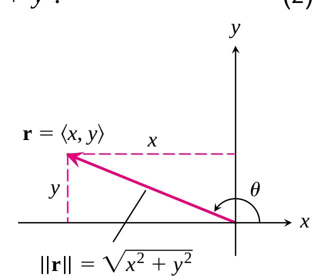
The length and direction of any vector $\overrightarrow{AB}$ equivalent to $\mathbf{r} = \langle x, y \rangle$, where $x = b_1 - a_1$ and $y = b_2 - a_2$, are the same as the length and direction of $\mathbf{r}$. Hence, we may calculate the length and direction of $\overrightarrow{AB}$ in terms of x and y.

Length and Direction of a Vector

Let $\mathbf{r} = \langle x, y \rangle$ be a vector from the origin. The **length** of $\mathbf{r}$, and of any vector equivalent to $\mathbf{r}$, is

$$\|\mathbf{r}\| = \|\langle x, y \rangle\| = \sqrt{x^2 + y^2}. \tag{2}$$

The **direction** of $\mathbf{r} = \langle x, y \rangle$, and of any vector equivalent to $\mathbf{r}$, is the angle θ through which the positive x-axis must be rotated counterclockwise to align with $\mathbf{r}$. If $\mathbf{r} = \langle 0, 0 \rangle$, we assign no direction to $\mathbf{r}$. See accompanying figure.



The length $\|\mathbf{r}\|$ and direction θ of a representative vector $\mathbf{r}$.

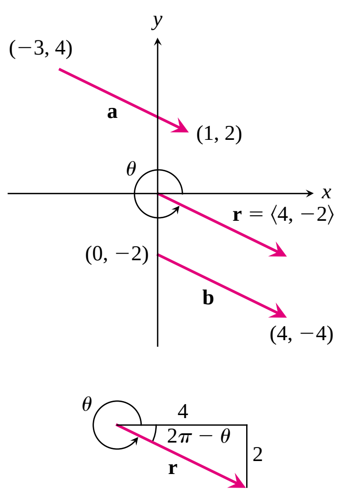


FIGURE 3.13 Equivalent vectors **a**, **b**, and **r**.

We shall often denote a vector from an initial point P to a terminal point Q by a single boldface letter such as $\mathbf{a}$ rather than $\overrightarrow{PQ}$. Figure 3.13 shows, for example, a vector $\mathbf{a}$ from $(-3, 4)$ to $(1, 2)$.

EXAMPLE 1 Figure 3.13 shows vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{r}$. Show that these vectors are equivalent. Calculate their common length and direction.

Solution

The vectors **a**, **b**, and **r** are equivalent because, by applying (1),

$$1 - (-3) = 4 - 0 = 4 - 0 = 4 \quad \text{and}$$
$$2 - 4 = -4 - (-2) = -2 - 0 = -2.$$

The common length and direction of these vectors are the length and direction of the position vector $\mathbf{r} = \langle 4, -2 \rangle$. The length of **r** is

$$\|\mathbf{r}\| = \sqrt{4^2 + (-2)^2} = \sqrt{20} \approx 4.47.$$

The direction of **r** is the angle θ shown in Fig. 3.13. Perhaps the easiest way to calculate θ is to calculate the acute angle $2\pi - \theta$ shown in the diagram at the bottom of the figure. The tangent of this angle is $2/4 = 0.5$. Hence

$$\theta = 2\pi - \arctan 0.5 = 2\pi - 0.46364\cdots \approx 5.8.$$

Vector Addition and the Parallelogram Law

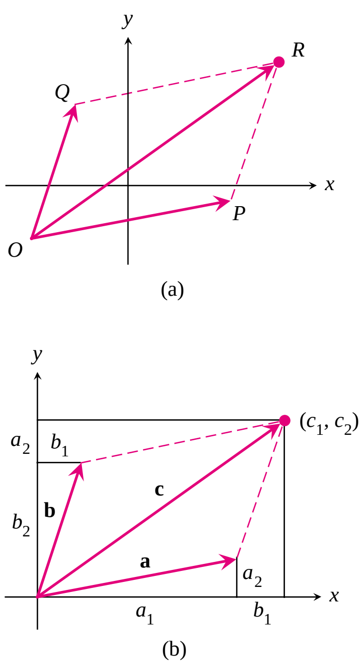


FIGURE 3.14 (a) Adding vectors by applying the parallelogram law; (b) adding position vectors **a** and **b** arithmetically.

Vector addition can be motivated by a physical experiment. Consider two forces acting at a point O of the plane. Figure 3.14(a) shows these forces as the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. The lengths of the vectors are proportional to the magnitudes of the two forces, and their directions are the same as those of the two forces. It can be shown by experiment that the combined effect of the two forces acting at O is equivalent to the single force corresponding to the vector $\overrightarrow{OR}$, which is the diagonal of the parallelogram with sides $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. This is sometimes called the *parallelogram law*.

The vector addition $\overrightarrow{OP} + \overrightarrow{OQ}$ shown in Fig. 3.14(a) is based on the physical/geometric parallelogram law. For most applications involving vector addition it is more convenient to form the sum $\overrightarrow{OP} + \overrightarrow{OQ}$ arithmetically. We do this by defining the sum $\mathbf{a} + \mathbf{b}$ of the position vectors **a** and **b** equivalent to $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. We show that this definition is consistent with the parallelogram law by showing that $\mathbf{a} + \mathbf{b}$ is equivalent to the vector $\overrightarrow{OP} + \overrightarrow{OQ}$.

Vector Addition

The **sum** of the vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is the vector

$$\mathbf{a} + \mathbf{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle. \tag{3}$$

Figure 3.14(b) gives a geometric interpretation of this definition. Letting **c** be the diagonal vector of the parallelogram formed by **a** and **b**, note that the sum of the segments with lengths a_1 and b_1 is equal to the segment of length c_1, where c_1 is the first coordinate of **c**. Similarly, the sum of the segments with lengths a_2 and b_2 is equal to the segment of length c_2. Thus,

$$\begin{aligned}\mathbf{c} &= \langle c_1, c_2 \rangle \\ &= \langle a_1 + b_1, a_2 + b_2 \rangle \\ &= \langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle = \mathbf{a} + \mathbf{b}.\end{aligned}$$

Hence, adding position vectors by applying (3) is consistent with adding vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in Fig. 3.14(a) by applying the parallelogram law.

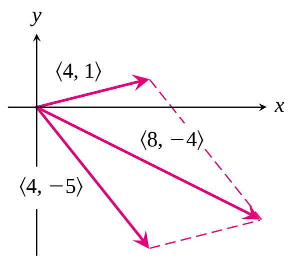


FIGURE 3.15 Adding position vectors.

EXAMPLE 2 Calculate the sum of vectors $\langle 4, 1\rangle$ and $\langle 4, -5\rangle$.

Solution

From the definition of vector addition,

$$\langle 4, 1\rangle + \langle 4, -5\rangle = \langle 4 + 4, 1 + (-5)\rangle = \langle 8, -4\rangle.$$

Figure 3.15 shows the given vectors as well as their sum. The vector $\langle 8, -4\rangle$ is a diagonal of the parallelogram determined by $\langle 4, 1\rangle$ and $\langle 4, -5\rangle$.

The consistency mentioned earlier of the physical/geometric sum $\overrightarrow{OP} + \overrightarrow{OQ}$ and the arithmetic sum $\mathbf{a} + \mathbf{b}$ of position vectors $\mathbf{a}$ and $\mathbf{b}$ equivalent to $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ tends to blur the distinction between equivalent vectors and equal vectors. For example, suppose that the points O, P, and Q in Fig. 3.14(a) have coordinates (o_1, o_2), (p_1, p_2), and (q_1, q_2). How can we calculate the coordinates (r_1, r_2) of R in terms of the coordinates of O, P, and Q? If we write $\overrightarrow{OP} = \langle p_1 - o_1, p_2 - o_2\rangle$, $\overrightarrow{OQ} = \langle q_1 - o_1, q_2 - o_2\rangle$, and $\overrightarrow{OR} = \langle r_1 - o_1, r_2 - o_2\rangle$, then because $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$,

$$\begin{aligned}\langle r_1 - o_1, r_2 - o_2\rangle &= \langle p_1 - o_1, p_2 - o_2\rangle + \langle q_1 - o_1, q_2 - o_2\rangle \\ &= \langle p_1 + q_1 - 2o_1, p_2 + q_2 - 2o_2\rangle.\end{aligned}$$

Hence, the coordinates r_1 and r_2 of R are

$$\begin{aligned}r_1 &= p_1 + q_1 - o_1 \\ r_2 &= p_2 + q_2 - o_2.\end{aligned}$$

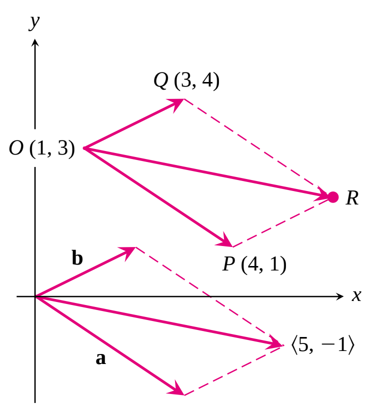


FIGURE 3.16 Adding vectors.

EXAMPLE 3 Let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ be vectors, where the points O, P, and Q are $(1, 3)$, $(4, 1)$, and $(3, 4)$. See Fig. 3.16. Calculate the length and direction of the sum $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$. Also, calculate the coordinates of R.

Solution

The sum of vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ can be found by adding position vectors $\mathbf{a}$ and $\mathbf{b}$ equivalent to $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. The sum $\mathbf{a} + \mathbf{b}$ has the same length and direction as the vector $\overrightarrow{OR}$. From Fig. 3.16,

$$\mathbf{a} = \langle 4 - 1, 1 - 3\rangle = \langle 3, -2\rangle \quad \text{and} \quad \mathbf{b} = \langle 3 - 1, 4 - 3\rangle = \langle 2, 1\rangle.$$

From the definition of vector addition,

$$\mathbf{a} + \mathbf{b} = \langle 3, -2\rangle + \langle 2, 1\rangle = \langle 5, -1\rangle.$$

This vector is equivalent to the vector $\overrightarrow{OR}$ and, therefore, has the same length and direction. It follows that

$$\|\overrightarrow{OR}\| = \sqrt{5^2 + (-1)^2} = \sqrt{26}.$$

The direction θ of $\overrightarrow{OR}$ can be found as in Example 1:

$$\theta = 2\pi - \arctan(1/5) \approx 6.09.$$

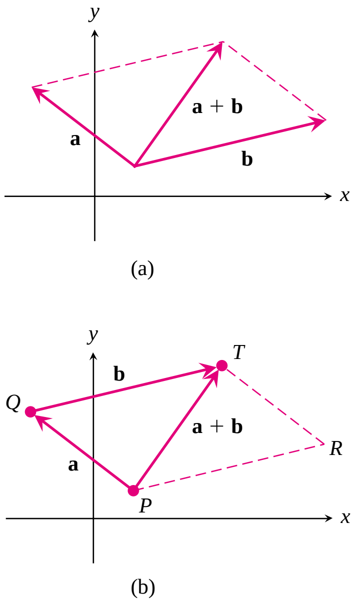


FIGURE 3.17 (a) Vector addition interpreted as, for example, two forces acting at a single point; (b) vector addition interpreted as the sum of successive displacements.

The coordinates r_1 and r_2 of R can be found from the fact that $\overrightarrow{OR}$ is equivalent to $\mathbf{a} + \mathbf{b} = \langle 5, -1 \rangle$, which means that $r_1 - 1 = 5$ and $r_2 - 3 = -1$. Hence, the coordinates of R are $(6, 2)$.

Figure 3.17(a) shows a graphical representation of the sum of **a** and **b**, with the tails of **a**, **b**, and **a** + **b** at the same point. In Fig. 3.17(b), the vectors labeled **a**, **b**, and **a** + **b** are equivalent to the vectors in Fig. 3.17(a), but **a** and **b** are arranged "tip-to-tail." This is useful if, for example, **a** and **b** represent successive displacements of an object. With this interpretation, the sum **a** + **b** represents a single displacement, equivalent to first displacing an object from P to Q and then from Q to T. The sum **a** + **b** is also equivalent to first displacing an object from P to R and then from R to T.

Scalar Multiplication

When we work with both vectors and real numbers in the same context, real numbers are often called **scalars**.

Scalar Multiplication

The **product** of a scalar s and a vector $\mathbf{r} = \langle x, y \rangle$ is

$$s\mathbf{r} = s\langle x, y \rangle = \langle sx, sy \rangle. \tag{4}$$

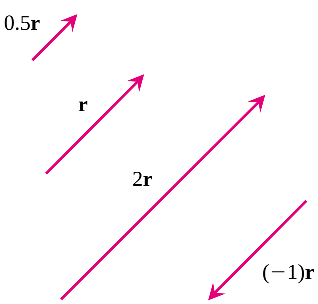


FIGURE 3.18 Multiplying a vector **r** by scalars 0.5, 2, and -1.

Figure 3.18 shows a vector **r** and several scalar multiples of **r**. Because

$$\begin{aligned} \|s\mathbf{r}\| &= \|\langle sx, sy \rangle\| = \sqrt{s^2x^2 + s^2y^2} \\ &= |s|\sqrt{x^2 + y^2} = |s| \cdot \|\mathbf{r}\|, \end{aligned}$$

the length of the vector $s\mathbf{r}$ is $|s|$ times the length of **r**. If $s > 0$, the vector $s\mathbf{r}$ points in the same direction as **r**; if $s < 0$, $s\mathbf{r}$ points in the opposite direction.

Subtracting Vectors

The **difference** $\mathbf{a} - \mathbf{b}$ of the vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is the vector $\mathbf{a} + (-1)\mathbf{b}$. Hence,

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \langle a_1, a_2 \rangle + (-1)\langle b_1, b_2 \rangle \\ &= \langle a_1, a_2 \rangle + \langle -b_1, -b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle. \end{aligned} \tag{5}$$

Figure 3.19(a) shows vectors **a**, **b**, $(-1)\mathbf{b}$, and $\mathbf{a} + (-1)\mathbf{b}$. Adding the vectors **a** and $(-1)\mathbf{b}$ by the parallelogram law gives the vector $\mathbf{a} + (-1)\mathbf{b} = \mathbf{a} - \mathbf{b}$. From Fig. 3.19(a), it is clear that the vectors **a** + **b** and **a** − **b** are the two diagonals of the parallelogram formed by the vectors **a** and **b**. Fig. 3.19(b) shows both the sum and difference of the vectors **a** and **b**.

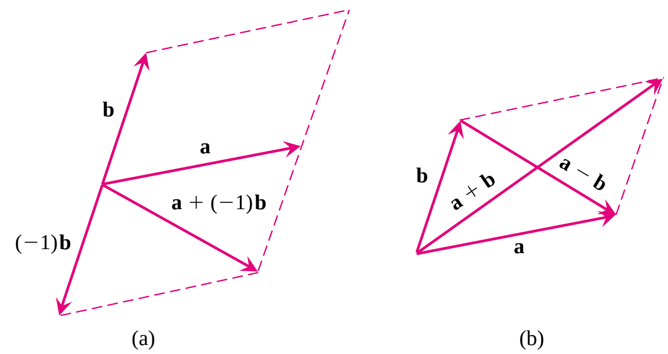


FIGURE 3.19 The sum and difference of vectors **a** and **b** are the two diagonals of the parallelogram formed by **a** and **b**.

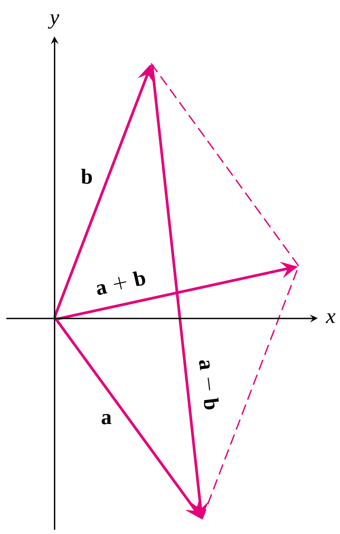


FIGURE 3.20 The sum and difference of vectors **a** and **b** are the two diagonals of the parallelogram formed by **a** and **b**.

EXAMPLE 4 If $\mathbf{a} = \langle 3, -4\rangle$ and $\mathbf{b} = \langle 2, 5\rangle$, calculate $\|\mathbf{a}\|$, $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, and $-2\mathbf{a} + 7\mathbf{b}$. Sketch the vectors **a**, **b**, $\mathbf{a} + \mathbf{b}$, and $\mathbf{a} - \mathbf{b}$.

Solution

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5 \\ \mathbf{a} + \mathbf{b} &= \langle 3, -4\rangle + \langle 2, 5\rangle = \langle 5, 1\rangle \\ \mathbf{a} - \mathbf{b} &= \langle 3, -4\rangle - \langle 2, 5\rangle = \langle 1, -9\rangle \\ -2\mathbf{a} + 7\mathbf{b} &= -2\langle 3, -4\rangle + 7\langle 2, 5\rangle = \langle -6, 8\rangle + \langle 14, 35\rangle = \langle 8, 43\rangle. \end{aligned}$$

Figure 3.20 shows the vectors **a**, **b**, $\mathbf{a} + \mathbf{b}$, and $\mathbf{a} - \mathbf{b}$.

The i and j Vectors

The simple vectors $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$ provide an alternative way to express a vector. If $\mathbf{r} = \langle x, y\rangle$, then

$$\mathbf{r} = \langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle = x\langle 1, 0\rangle + y\langle 0, 1\rangle = x\mathbf{i} + y\mathbf{j}. \tag{6}$$

Thus we see that the vectors $\langle x, y\rangle$ and $x\mathbf{i} + y\mathbf{j}$ are different ways of writing the same vector. The $x\mathbf{i} + y\mathbf{j}$ notation is widely used in engineering and physics.

Figure 3.21 shows the vectors **i** and **j**. Also shown are the vectors

$$\langle 2, 3\rangle = 2\langle 1, 0\rangle + 3\langle 0, 1\rangle = 2\mathbf{i} + 3\mathbf{j}$$

and

$$\langle -2, -1\rangle = (-2)\langle 1, 0\rangle - \langle 0, 1\rangle = -2\mathbf{i} - \mathbf{j}.$$

In what follows we use either $\langle a_1, a_2\rangle$ or $a_1\mathbf{i} + a_2\mathbf{j}$ for a vector **a**.

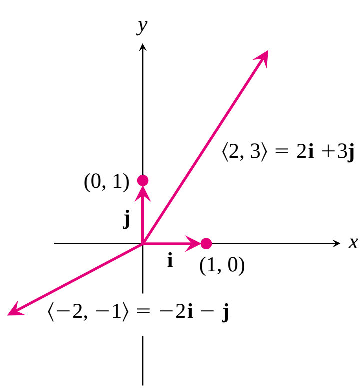


FIGURE 3.21 Writing vectors $\langle 2, 3\rangle$ and $\langle -2, -1\rangle$ in terms of the vectors **i** and **j**.

EXAMPLE 5 Figure 3.22 shows the forces **a** and **b** exerted by two tugboats on a coal barge. The magnitudes of these forces are $\|\mathbf{a}\| = 90$ kilonewtons (approximately 10 tons) and $\|\mathbf{b}\| = 70$ kilonewtons (approximately 8 tons). Find the magnitude and direction of the *resultant force* on the barge, that is, the vector sum of the forces **a** and **b**.

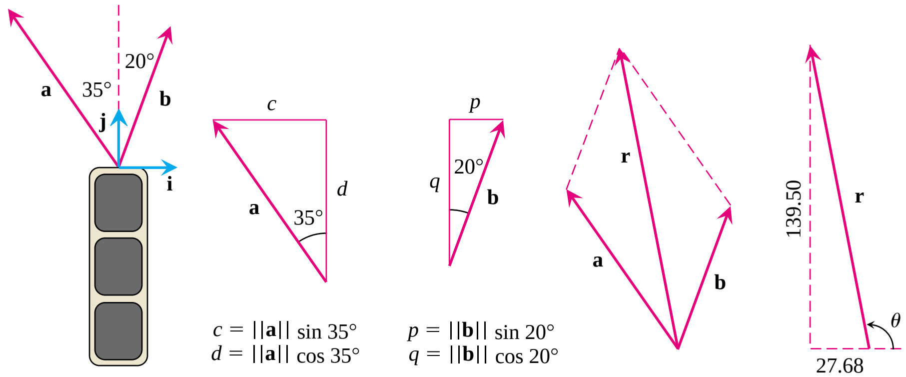


FIGURE 3.22 Forces **a** and **b** on a barge.

Solution

It is natural to think of the barge and the two vectors **a** and **b** in a coordinate system whose origin is at the point on the barge where **a** and **b** act, and the positive *x*- and *y*-axes extending to the right and upward, respectively. The **i** and **j** vectors will be as shown at the extreme left of Fig. 3.22. The vectors **a** and **b** are shown separately in the center of Fig. 3.22. The sides *c*, *d*, *p*, and *q* of the triangles can be calculated using right-triangle trigonometry. Adjusting the signs to fit the directions of **a** and **b**,

$$\mathbf{a} = -c\mathbf{i} + d\mathbf{j}$$
$$\mathbf{b} = p\mathbf{i} + q\mathbf{j}.$$

Hence,

$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \mathbf{b}\\ &= -c\mathbf{i} + d\mathbf{j} + p\mathbf{i} + q\mathbf{j} = (-c + p)\mathbf{i} + (d + q)\mathbf{j}\\ &= (-90 \sin 35^\circ + 70 \sin 20^\circ)\mathbf{i} + (90 \cos 35^\circ + 70 \cos 20^\circ)\mathbf{j}\\ &\approx (-27.68)\mathbf{i} + 139.50\mathbf{j}.\end{aligned}$$

The length or magnitude of the resultant force is

$$\sqrt{(-27.68)^2 + (139.50)^2} = 142.22 \text{ kilonewtons.}$$

Referring to the sketch on the extreme right of the figure, the direction θ of the resultant force is

$$\theta = 180^\circ - \arctan(139.50/27.68) \approx 180^\circ - 78.78^\circ = 101.22^\circ.$$

Unit Vectors

A vector **u** is a unit vector if $\|\mathbf{u}\| = 1$, that is, if the length of **u** is 1. Given a vector $\mathbf{r} \neq \mathbf{0}$, a unit vector **u** in the same direction as **r** is

$$\mathbf{u} = \frac{1}{\|\mathbf{r}\|}\mathbf{r},$$

For example, if $\mathbf{r} = \langle 4, -3 \rangle$, then $\|\mathbf{r}\| = \sqrt{16 + 9} = 5$ and

$$\mathbf{u} = \frac{1}{\|\mathbf{r}\|}\mathbf{r} = \frac{1}{5}\langle 4, -3 \rangle = \langle 4/5, -3/5 \rangle.$$

It is easy to check that $\|\mathbf{u}\| = \sqrt{(4/5)^2 + (-3/5)^2} = 1$.

To find a unit vector in a given direction, we use the standard definition of *sine* and *cosine*. Referring to Fig. 3.23, suppose we want to find a vector in the direction θ. If we place an angle θ in standard position (vertex at the origin and initial side along the positive x-axis), then the coordinates of the point of intersection of the terminal side of the angle with the unit circle are $(\cos\theta, \sin\theta)$. Hence, the position vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ has unit length and direction θ.

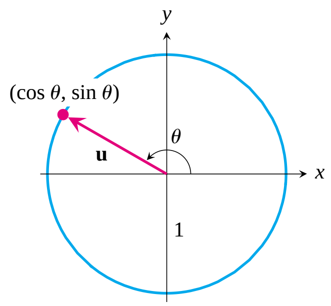


FIGURE 3.23 Corresponding to the angle θ are the point $(\cos\theta, \sin\theta)$ on the unit circle and the unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$.

Unit Vector in a Given Direction

A unit vector $\mathbf{u}$ in the direction θ is

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle. \tag{7}$$

EXAMPLE 6 Find a unit vector in the direction $\theta = 5\pi/6$ or 150°. Also, find a vector $\mathbf{F}$ with length 3.5 and direction $\theta = 0.3$.

Solution

Figure 3.23 shows an angle $\theta \approx 150°$. From (7), a unit vector in this direction is

$$\mathbf{u} = \langle \cos 150°, \sin 150° \rangle = \left\langle -\sqrt{3}/2, 1/2 \right\rangle.$$

To find the vector with length 3.5 and direction $\theta = 0.3$, we find a unit vector $\mathbf{u}$ in this direction and then multiply $\mathbf{u}$ by 3.5.

$$\mathbf{u} = \langle \cos 0.3, \sin 0.3 \rangle \approx \langle 0.95534, 0.29552 \rangle.$$

Multiplying this vector by the scalar 3.5,

$$\mathbf{F} = 3.5\mathbf{u} \approx \langle 3.34368, 1.03432 \rangle.$$

As a check, note that $\|\mathbf{F}\| = \sqrt{3.34368^2 + 1.03432^2} \approx \sqrt{12.25001} \approx 3.50000$.

Displacements The displacement of objects along a line or in a plane is often described by vectors. If an object at the point (x, y) is displaced a distance r and in the direction θ, we may describe this displacement by the vector $\mathbf{a} = \langle a_1, a_2 \rangle = r\langle \cos\theta, \sin\theta \rangle$ having magnitude r and direction θ. The coordinates of the object after displacement are $(x + a_1, y + a_2)$. Figure 3.24 shows a representative displacement $\mathbf{a}$ and shows how the "after" coordinates (X, Y) of the object may be calculated from its "before" coordinates (x, y). If $\langle x, y \rangle$ and $\langle X, Y \rangle$ are the position vectors of the object before and after displacement, then directly from the figure we see that

$$\langle X, Y \rangle = \langle x, y \rangle + \mathbf{a}.$$

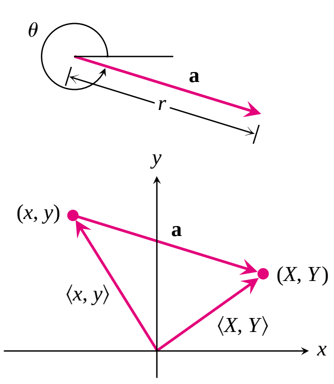


FIGURE 3.24 The displacement $\mathbf{a}$ of an object from (x, y) to (X, Y).

If an object is subject to successive displacements, the coordinates of its final position can be calculated by adding the several displacement vectors to obtain a single, equivalent displacement.

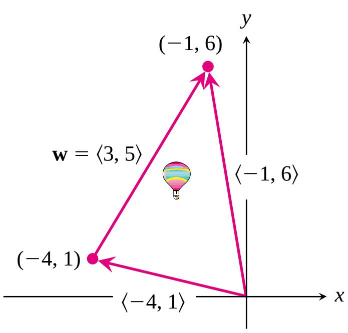


FIGURE 3.25 The displacement $\mathbf{a} = \langle 3, 5\rangle$ of an object initially at $(-4, 1)$.

EXAMPLE 7 A hot-air balloon was first sighted at a point with map coordinates $(-4, 1)$, where distances are measured in kilometers. Assuming that the balloon maintains its altitude and the prevailing breeze displaces it by the vector $\mathbf{w} = \langle 3, 5\rangle$, what is its final position?

Solution

Figure 3.25 shows the balloon at its initial position $(-4, 1)$ and the displacement vector $\mathbf{w} = \langle 3, 5\rangle$. The coordinates (X, Y) of the final position of the balloon satisfy

$$\langle X, Y\rangle = \langle -4, 1\rangle + \mathbf{w} = \langle -4, 1\rangle + \langle 3, 5\rangle = \langle -1, 6\rangle.$$

Hence, $(X, Y) = (-1, 6)$.

We summarize the properties of vector addition and scalar multiplication. Each of these can be proved from the properties of real numbers and the definitions of vector addition and scalar multiplication. See, for example, Exercises 48 and 49.

Properties of Vector Addition and Scalar Multiplication

Let $\mathbf{0} = \langle 0, 0\rangle$ be the zero vector. For vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and scalars p and q,

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $1\mathbf{a} = \mathbf{a}$
6. $(p + q)\mathbf{a} = p\mathbf{a} + q\mathbf{a}$
7. $(pq)\mathbf{a} = p(q\mathbf{a})$
8. $p(\mathbf{a} + \mathbf{b}) = p\mathbf{a} + p\mathbf{b}$

Exercises 3.2

Exercises 1–6: Show that $\overrightarrow{PQ}$ and $\overrightarrow{ST}$ are equivalent, find a position vector $\mathbf{r}$ to which both are equivalent, and calculate the common lengths of these vectors. Sketch $\mathbf{r}$, $\overrightarrow{PQ}$ and $\overrightarrow{ST}$.

1. $P(1, 1)$, $Q(4, -1)$, $S(7, 5)$, and $T(10, 3)$
2. $P(-1, -2)$, $Q(-4, -1)$, $S(5, -1)$, and $T(2, 0)$
3. $P(3, -2/3)$, $Q(5, -2)$, $S(-6, 16/3)$, and $T(-4, 4)$
4. $P(3, 10)$, $Q(1, 4)$, $S(-2, -5)$, and $T(-4, -11)$
5. $P(-2.5, 3.0)$, $Q(-0.8, 1.1)$, $S(0.0, -1.3)$, and $T(1.7, -3.2)$
6. $P(2.5, 0.9)$, $Q(1.1, -0.3)$, $S(2.3, -2.2)$, and $T(0.9, -3.4)$

Exercises 7–10: Find $\mathbf{a} + \mathbf{b}$. Use these vectors and their sum to illustrate the parallelogram law.

7. $\mathbf{a} = \langle 2, 5\rangle$ and $\mathbf{b} = \langle 5, 2\rangle$
8. $\mathbf{a} = \langle 2, 6\rangle$ and $\mathbf{b} = \langle 4, 1\rangle$

9. $\mathbf{a} = \langle -3, 3\rangle$ and $\mathbf{b} = \langle 1, 4\rangle$

10. $\mathbf{a} = \langle -3, -3\rangle$ and $\mathbf{b} = \langle 1, -5\rangle$

Exercises 11–14: Find a position vector equivalent to $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$. Also find the coordinates of R. Sketch $\overrightarrow{OR}$, $\overrightarrow{OP}$, and $\overrightarrow{OQ}$.

11. $O = (1, 1)$, $P = (4, 2)$, $Q = (2, 7)$

12. $O = (-1, -2)$, $P = (4, 1)$, $Q = (5, 5)$

13. $O = (-2.5, 3.0)$, $P = (-0.8, 1.1)$, $Q = (0.0, 3.3)$

14. $O = (2.5, 0.9)$, $P = (1.1, -0.3)$, $Q = (2.3, -2.2)$

Exercises 15–22: Calculate $\|\mathbf{a}\|$, $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, and $h\mathbf{a} + k\mathbf{b}$.

15. $\mathbf{a} = \langle 5, -12\rangle$, $\mathbf{b} = \langle 1, 4\rangle$, $h = 1$, and $k = 2$

16. $\mathbf{a} = \langle -24, 7\rangle$, $\mathbf{b} = \langle 2, -1\rangle$, $h = 2$, and $k = 1$

17. $\mathbf{a} = -4\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j}$, $h = -2$, $k = 1$

18. $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 7\mathbf{j}$, $h = -2$, $k = 2$

19. $\mathbf{a} = \langle 1, 6\rangle$, $\mathbf{b} = \langle 1, -4\rangle$, $h = 0.5$, $k = -0.5$

20. $\mathbf{a} = \langle -1/3, 2/5\rangle$, $\mathbf{b} = \langle 2/3, -3/4\rangle$, $h = 1$, $k = 2$

21. $\mathbf{a} = 1.5\mathbf{i} - 2.4\mathbf{j}$, $\mathbf{b} = 0.3\mathbf{i} + 2.2\mathbf{j}$, $h = 1.6$, $k = -3.4$

22. $\mathbf{a} = -11.5\mathbf{i} - 12.4\mathbf{j}$, $\mathbf{b} = 9.6\mathbf{i} + 5.9\mathbf{j}$, $h = 0.5$, $k = -0.7$

Exercises 23–28: Find a unit vector in the direction θ.

23. $\theta = \pi/6$

24. $\theta = \pi/3$

25. $\theta = 210°$

26. $\theta = 240°$

27. $\theta = 5.0$

28. $\theta = 4.5$

29. The mass of the (plastic) stoplight in the figure is approximately 4.5 kilograms. Due to the Earth's attraction, the stoplight exerts a vertical force of 44.1 newtons at the point where the cables are attached. Find the forces the cables exert on the stoplight. *Hint:* The resultant of the cable forces must be equal and opposite (in direction) to the force of 44.1 newtons.

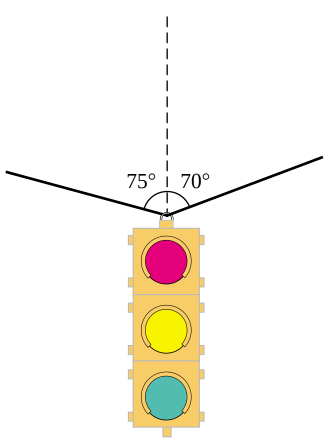


Figure for Exercise 29.

30. As shown in the figure, a tightrope walker has reached the middle of a stretched cable, resulting in a vertical force of 700 newtons at the center point. Find the forces the cables must exert to balance the performer. *Hint:* The resultant of the cable forces must be equal and opposite (in direction) to the force of 700 newtons.

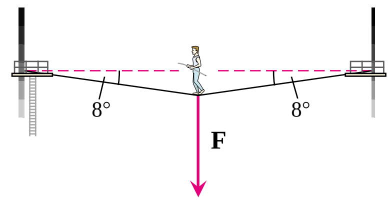


Figure for Exercise 30.

31. Because of bears, two campers have suspended their food pack from ropes attached to two trees, as shown in the figure, resulting in a vertical force of 200 newtons on the rope. Find the forces the ropes must exert to balance the food pack. *Hint:* The resultant of the rope forces must be equal and opposite (in direction) to the vertical force of 200 newtons.

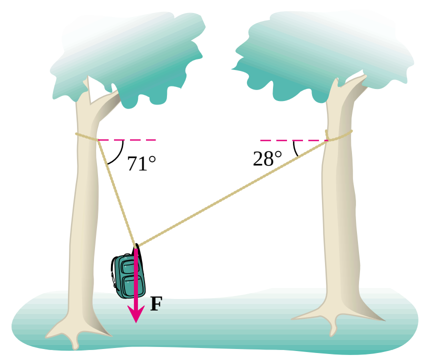


Figure for Exercise 31.

32. What force **F** will just balance the forces **A**, **B**, and **C** shown in the figure, where the magnitudes of these forces are 300 newtons, 350 newtons, and 500 newtons and the angles they make with the dotted line are 30°, 25°, and 20°, respectively?

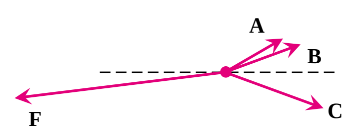


Figure for Exercise 32.

33. Let $\mathbf{r} = \langle 3, 5\rangle$. Find a vector in the same direction as $\mathbf{r}$ but with length 8.

34. Let $\mathbf{r} = \langle 3, 2\rangle$. Find a vector in the same direction as $\mathbf{r}$ but with length 7.

35. Let A, B, and C be the points $(1, 4)$, $(2, 6)$, and $(-1, 3)$. Find a point D so that $\overrightarrow{CD}$ is twice as long as $\overrightarrow{AB}$ and in the opposite direction.

36. Let A, B, and C be the points $(-3, 1)$, $(3, -4)$, and $(5, 3)$. Find a point D so that $\overrightarrow{CD}$ is twice as long as $\overrightarrow{AB}$ and in the opposite direction.

37. Let $P = (3, 4)$, $Q = (8, 6)$, and $S = (5, 7)$ be points in a plane. Find the coordinates of the fourth vertex of the parallelogram whose sides are $\overrightarrow{PQ}$ and $\overrightarrow{PS}$.

38. Let $P = (1, -3)$, $Q = (-1, -5)$, and $S = (4, -2)$ be points in a plane. Find the coordinates of the fourth vertex of the parallelogram whose sides are $\overrightarrow{PQ}$ and $\overrightarrow{PS}$.

39. Find the coordinates of an object that has been displaced from the point $(-4, 9)$ by the vector $\langle 4, -5\rangle$.

40. Find the coordinates of an object that has been displaced from the point $(0, 4)$ by the vector $\langle -2, -5\rangle$.

41. Find the coordinates of an object that has been displaced from the point $(5, 5)$ a distance of 10 units and in the direction $\theta = 30°$.

42. Find the coordinates of an object that has been displaced from the point $(0, -7)$ a distance of 13 units and in the direction $\theta = 135°$.

43. An object at $(0, 4)$ is displaced to Q by $2\mathbf{i} + 5\mathbf{j}$ and then to T by $-12\mathbf{i} + 13\mathbf{j}$. Calculate the single equivalent displacement and the coordinates of T.

44. An object at $(-4, 9)$ is displaced to Q by $\langle -4, -5\rangle$ and then to T by $\langle 1, 1\rangle$. Calculate the single equivalent displacement and the coordinates of T.

45. Find the coordinates of the object initially at $(1, 1)$ and subjected to successive displacements $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, where the magnitudes and directions of these displacements are 10, 5, and 7 and 330°, 60°, and 180°, respectively.

46. Find the coordinates of the object initially at $(-10, 15)$ and subjected to successive displacements $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, where the magnitudes and directions of these displacements are 12, 5, and 15 and 80°, 170°, and 280°, respectively.

47. Find a position vector $\mathbf{r}$ parallel to a line with slope m. What is the slope of a line parallel to the vector $\langle a, b\rangle$, where $a \neq 0$?

48. Show that if $\mathbf{a}$ is a vector and p and q scalars, then $p(q\mathbf{a}) = (pq)\mathbf{a}$. *Hint:* Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$.

49. Show that vector addition is associative: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, then $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$. *Hint:* Let $\mathbf{a} = \langle a_1, a_2\rangle$, $\mathbf{b} = \langle b_1, b_2\rangle$, and $\mathbf{c} = \langle c_1, c_2\rangle$.

50. The distance and bearing of Site A on the other side of a lake from Base Camp are required. Measuring all bearings counterclockwise from the positive x-axis, a survey team travels 10 km from Base Camp to Station S1 on a bearing of 30°; 15 km from S1 to S2 on a bearing of 110°; and, finally, 12 km on a bearing of 160°. Determine the distance and bearing of Site A from Base Camp.

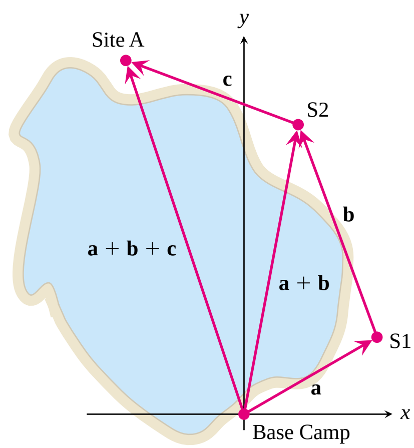


Figure for Exercise 50.

51. Show that for all vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

This inequality is called the *triangle inequality*. Use a sketch and a few sentences to explain why this name is appropriate.

3.3 Parametric Equations

The orbit of Mars about the sun is an ellipse with one of its two focii at the sun. This ellipse can be described by an equation of the form

$$\frac{(x + c)^2}{a^2} + \frac{y^2}{b^2} = 1,$$