

in the body. However, the saline does not contain red blood cells, so cells lost during surgery are not replaced. Normally human blood is about 40% red cells by volume. If the red cell level is too low at the end of surgery, the patient may not survive. To lessen the red cell loss, some of a patient's blood may be removed before surgery and replaced with saline. The blood is then returned to the patient after the surgery. This process, called acute normovolemic hemodilution (ANH), has the effect of decreasing the amount of red cells lost during surgery.

We can model the patient's red cell levels during surgery and use this model to assess the effect of ANH. We make the following assumptions:

1. During surgery 2.5 liters of blood is lost, and the loss occurs at a constant rate.
2. Saline is administered at a rate equal to the rate of blood loss, keeping the combined volume of the blood and saline at 5 liters.
3. The saline mixes immediately with the blood in the body, which means that the red cell concentration is uniform throughout the blood.

Suppose that the surgery lasts T minutes. By assumption 1, the blood is lost at a rate of $(2.5/T)$ liters/minute. Let $R(t)$ be the volume of red cells at time t . Then at time t each liter of blood contains $R(t)/5$ liters of red cells. It follows that the rate of loss of the red cells is

$$\begin{aligned} & \left(\frac{R(t)}{5} \text{ liters cells/liter blood} \right) \left(\frac{2.5}{T} \text{ liters blood/minute} \right) \\ &= \frac{1}{2T} R(t) \text{ liters cells/minute} \end{aligned}$$

Because this rate of change describes a decrease in the cell volume, it makes a negative contribution to the rate of change of the red cell level. Thus we have the differential equation

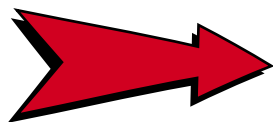
$$\frac{dR}{dt} = -\frac{1}{2T} R(t). \quad (6)$$

Before surgery or ANH the patient has $0.4 \cdot 5 = 2$ liters of red cells. If L liters of blood are removed and replaced with saline before the operation, then the patient enters surgery with $2 - 0.4 \cdot L$ liters of red cells. Thus if $t = 0$ corresponds to the start of surgery, then we have $R(0) = 2 - 0.4 \cdot L$. This starting (initial) condition together with (6) constitutes our model,

$$\frac{dR}{dt} = -\frac{1}{2T} R(t), \quad R(0) = 2 - 0.4 \cdot L. \quad (7)$$

In Exercise 7 we look at a solution to this model and investigate the effect of the ANH procedure.

This model is based on the article "Calculus in the Operating Room," by Pearl Toy and Stan Wagon, which appeared in the February 1995 issue of *The American Mathematical Monthly*.



Modeling Combat

In a 1916 paper entitled, "Aircraft in Warfare: The Dawn of the Fourth Arm," F. W. Lanchester developed a mathematical model for air combat. Over the years, Lanchester's model has been modified to describe many different kinds of battles, including combat between two ground forces. As an example, we model a simple

battle in a field between two armies, the good guys and the bad guys. Let $G(t)$ be the number of good guys at time t and $B(t)$ the number of bad guys at time t . Because battles often last from one to several days, we measure t in hours. During a battle, members of each army are killed by members of the other army. Armies also lose troops through noncombat losses (accidents, desertion, etc.). In addition, the size of an army can increase if reinforcements are available. We make some assumptions to help us describe the rate at which each army gains and loses troops:

1. Both armies move with equal ease about the battlefield. In particular, neither army has a positional advantage over the other (however, see Exercise 12).
2. For each army, combat losses are at a rate proportional to the size of the opposing force. The constant of proportionality reflects the efficiency of the opposing army.
3. An army suffers noncombat losses at a rate proportional to the size of the army.
4. At time t , reinforcements join the good guys at a rate $g(t)$ troops per hour, and join the bad guys at a rate of $b(t)$ troops per hour. (Because reinforcements usually come in sporadically, $b(t)$ and $g(t)$ are usually 0 for most times t .)

According to assumption 2, the good guys suffer combat losses at a rate $-kB(t)$, where the positive constant k is indicative of the efficiency of the bad guys. According to assumption 3, the good guys suffer noncombat losses at a rate $-\ell G(t)$, for some positive constant ℓ . Combining these rates of change with the reinforcement function in assumption 4, we have

$$\frac{dG}{dt} = -kB(t) - \ell G(t) + g(t). \quad (8)$$

Similarly, for the bad guys we have

$$\frac{dB}{dt} = -mG(t) - nB(t) + b(t). \quad (9)$$

To complete the model, we need to know the initial size (size at $t = 0$) of each army. Assume that the good guys start with $G(0) = G_0$ troops and the bad guys with $B(0) = B_0$ troops. Combining these initial conditions with (8) and (9), we have our combat model:

$$\begin{aligned} \frac{dG}{dt} &= -kB(t) - \ell G(t) + g(t) \\ \frac{dB}{dt} &= -mG(t) - nB(t) + b(t) \\ G(0) &= G_0 \\ B(0) &= B_0. \end{aligned}$$

Models of this type have been used to analyze ongoing battles and wars. Special adaptations of this model to guerrilla warfare were used to model aspects of the Vietnam War. Other applications have included studies of the Battle of the Bulge, the Battle of Iwo Jima, and the Battle of the Alamo.

Modeling a Bungee Jump

The Dangerous Sports Club was founded in late 1977 by a group of British thrill seekers anxious to add “excitement” to their lives. The club members periodically