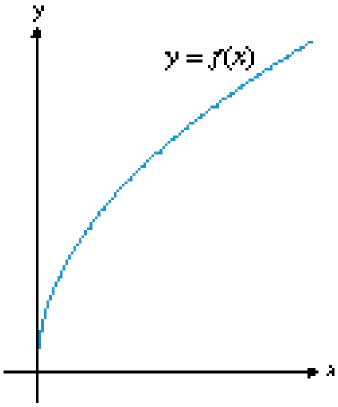

FIGURE 1.13 The graph of $y = f(x) = \sqrt{x}$.

Compositions and Graphs The graph of $y = f(x) = \sqrt{x}$ is shown in Fig. 1.13. Define a new function g by

$$g(x) = f(x - 3) = \sqrt{x - 3}. \tag{4}$$

What does the graph of $y = g(x)$ look like? We could produce the graph of g simply by plotting several points $(x, g(x))$ and then connecting the points with a "reasonable curve." This is how a calculator or CAS would produce the graph. On the other hand, the equations describing f and g are very closely related. The function g described in (4) is the composition of f with the linear function ℓ where $\ell(x) = x - 3$. It is not unusual to encounter two functions, one of which is obtained by composing the other with a linear function. When two functions are related in this way, the graph of one function can be obtained by translating, magnifying, shrinking, and/or reflecting the graph of the other.

Suppose we wish to plot the point $(a, g(a))$ on the graph of $y = g(x)$. Because $g(a) = f(a - 3)$, we see that the points $(a - 3, f(a - 3))$ and $(a, g(a))$ have the same y-coordinate, so they are at the same distance above or below the x-axis. See Fig. 1.14. In addition, the point $(a, g(a))$ is three units to the right of $(a - 3, f(a - 3))$. This means that the graph of $y = g(x)$ can be obtained by moving the graph of $y = f(x)$ three units to the right.

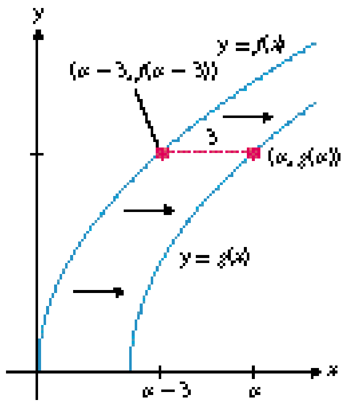

FIGURE 1.14 The points $(a - 3, f(a - 3))$ and $(a, g(a))$ have the same y-coordinates.

EXAMPLE 5 The graph of a function f is shown in Fig. 1.15.

a) Define g by $g(x) = 2f(x)$. Sketch the graph of $y = g(x)$.
b) Define h by $h(x) = f(2x)$. Sketch the graph of $y = h(x)$.

Solution

a) The domain of g is the same as the domain of f, so the graph of $y = g(x)$ lies above or below the same portion of the x-axis as the graph of $y = f(x)$. If a is in the domain of g, then $(a, g(a))$ is a point on the graph of $y = g(x)$ and $(a, f(a))$ is a point on the graph of $y = f(x)$. Because $g(a) = 2f(a)$, the y-coordinate of $(a, g(a))$ is twice the y-coordinate of the point $(a, f(a))$. It follows that the graph of $y = g(x)$ can be obtained by doubling the distance above or below the x-axis of each point on the graph of $y = f(x)$. See Fig. 1.16.

b) If a is in the domain of f, then $(a, f(a))$ is on the graph of $y = f(x)$. We also have

$$f(a) = f\left(2\frac{a}{2}\right) = h\left(\frac{a}{2}\right).$$

This means that the point $(a/2, h(a/2))$ is on the graph of $y = h(x)$ and has the same y-coordinate as the point $(a, f(a))$. Thus $(a/2, h(a/2))$ is the midpoint of the horizontal segment from $(a, f(a))$ to the y-axis. See Fig. 1.17. The graph of $y = h(x)$ can be obtained by halving the horizontal displacement from the y-axis of each point on the graph of $y = f(x)$.

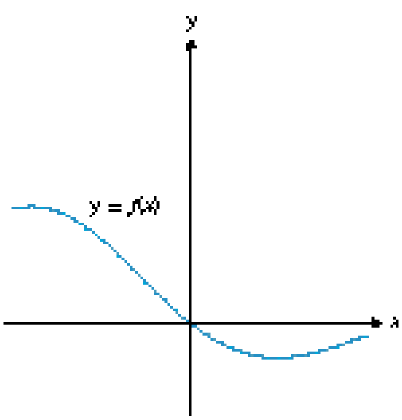

FIGURE 1.15 The graph of $y = f(x)$.

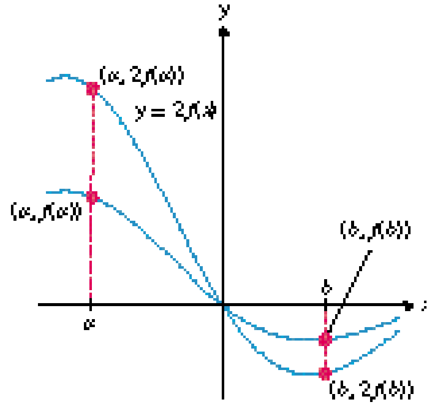

FIGURE 1.16 The graph of $y = 2f(x)$ is found by doubling the y-coordinate of each point on the graph of $y = f(x)$.

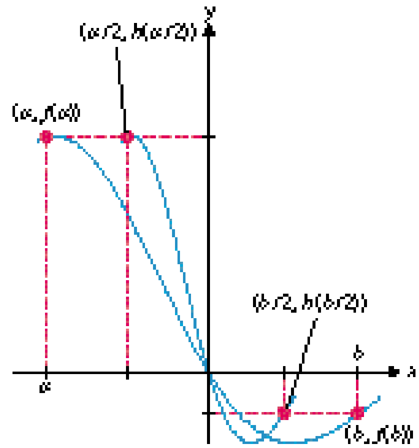

FIGURE 1.17 The graph of $y = f(2x)$ is found by halving the x-coordinate of each point on the graph of $y = f(x)$.

Graphical Composition When the graphs of two functions, f and g, are given, it is possible to use the graphs to find points on the graph of the composition $f \circ g$. We illustrate a method for doing this in the next example. In recent years this technique of "graphical composition" has been widely studied because it provides a means to illustrate chaos. See the project at the end of this chapter if you would like to explore some of the connections between compositions and chaos.

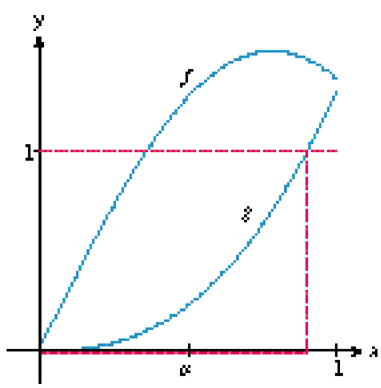

FIGURE 1.18 The domain of $f \circ g$ includes those x for which $0 \leq g(x) \leq 1$.

EXAMPLE 6 Figure 1.18 shows the graphs of two functions f and g, both of which have domain $[0, 1]$. A real number a is marked on the x-axis. Plot the point with coordinates $(a, (f \circ g)(a))$.

Solution

We first determine the domain of $f \circ g$. Because the domain of f is $[0, 1]$, the expression $f(g(x))$ will make sense if and only if $0 \leq g(x) \leq 1$. Thus any x values for which $g(x) > 1$ or $g(x) < 0$ are not in the domain of $f \circ g$. In Fig. 1.18 we see that $0 < g(a) < 1$, so a is in the domain of $f \circ g$. We now use the following four steps to find the point $(a, (f \circ g)(a))$ on the graph of the composition.

Step 1: From a on the horizontal axis draw a dashed line vertically to the graph of g. This line hits the graph at the point $(a, g(a))$.

Step 2: Think of the y-coordinate of the point $(a, g(a))$ as the "output" from the function g. To find $f(g(a))$, this output must become "input" for the function f. This is done by drawing a horizontal line from $(a, g(a))$ to the point $(g(a), g(a))$ on the line $y = x$.