

Exercises 8–11: Plot  $y$  as a function of  $x$  for  $-1 \leq x \leq 1$ . Using Table 2.4 as necessary, find  $\frac{dy}{dx}$ . Find the equation of the line tangent to the curve at the point corresponding to  $x = 0.5$ .

8.  $y = 2^x$

9.  $y = 3^x$

10.  $y = 4^x$

11.  $y = 7^x$

Exercises 12–31: Find the derivative. Use Table 2.4 as necessary.

12.  $y = e^{2x}$

13.  $y = e^{x/7}$

14.  $r = e^{-3t}$

15.  $u = e^{-t/2}$

16.  $f(x) = \sin(e^x)$

17.  $f(x) = \cos(2e^x)$

18.  $g(t) = e^{t^2}$

19.  $h(t) = e^{-t^2}$

20.  $z = e^{e^t}$

21.  $w = \sqrt{(e^x)^2 - e^x + 3}$

22.  $h(r) = r^4 e^r$

23.  $q(s) = 2^s e^{4s}$

24.  $s(t) = e^{2t} \cos(5t)$

25.  $R(t) = e^{-t} \sin(2t)$

26.  $v = \frac{e^u + 1}{e^u - 1}$

27.  $v = \frac{1}{e^u - e^{2u}}$

28.  $y = \tan(e^x + 1)$

29.  $y = \sec(e^x + 1)$

30.  $c(t) = \frac{e^t + e^{-t}}{2}$

31.  $s(t) = \frac{e^t - e^{-t}}{2}$

Exercises 32–35: Find an expression for  $y'$  if the equation defines  $y$  as a function of  $x$ .

32.  $(x + y)^2 = ye^x$

33.  $ye^{x+y} = 1$

34.  $e^x \tan(xy^2) - 2y = x + y$

35.  $e^{y^2} = xy$

36. Suppose you did not have a calculator but needed to find  $11^{1/5}$  accurate to within 0.001. Write a short paragraph describing how you would do this “by hand.” Write a similar paragraph describing how you would calculate  $11^{12/5}$  to an accuracy of 0.001.

37. In defining the exponential function to base  $b$  we offhandedly describe it as the function  $f$  defined by  $f(x) = b^x$ . Suppose that  $b = 11$  and  $x = \sqrt{5}$ . Write a short paragraph describing how you would calculate  $11^{\sqrt{5}}$  to within 0.001.

38. Verify any three of the  $C_b$  values in Table 2.4.

39. Suppose that in investigating the derivative of the exponential function, we decide that we want a base  $b$  so that

$$(b^x)' = 2 \cdot b^x.$$

What is the value of  $b$ ? Find the answer by numerical experimentation, and then by considering the equation  $(e^{bx})' = b \cdot e^{bx}$ .

40. When the sources of pollution are eliminated, a large polluted body of water eventually “cleanses” itself through natural processes. Observations have shown that during this cleansing period, the concentration of pollutants can be described by an exponential function of the form.

$$C(t) = C_0 e^{kt}.$$

In this expression,  $C(t)$  has units of parts per million (ppm) and is the concentration of pollutants at time  $t$ ,  $C_0$  is the concentration at time  $t = 0$ , and  $k$  is a constant. An environmentalist measures the pollutants in the lake at time  $t = 0$  and finds a concentration of 150 ppm. One year later, the concentration is 100 ppm.

- Find  $C_0$  and  $k$ .
- What will the concentration of pollutants be five years after the initial measurement?
- When will the concentration of pollutants be less than 10 ppm?

41. When money in a savings account earns interest at a rate of  $r$  percent per year, compounded daily, then the balance of the account  $T$  years after the money is deposited is well approximated by

$$A(t) = A_0 e^{0.01r \cdot T}$$

where  $A_0$  is the amount of the initial deposit. In the introduction to this section we stated that “a dollar invested at 4 percent interest per year compounded daily will provide your descendants 1000 years later with over ten thousand trillion dollars.” Verify this statement.

- The bacterium *Escherichia coli*, usually called *E. coli*, is found in the human digestive tract. Under ideal conditions each *E. coli* cell divides into two cells  $1/3$  hour after its own “birth.” Given that the mass of one *E. coli* cell is approximately  $5 \cdot 10^{-13}$  grams and assuming no deaths and ideal conditions, what is the mass of an *E. coli* colony after  $t$  hours, assuming that it started with one cell? What is the rate of change of the mass of the *E. coli* colony at time  $t = 10$  hours?
- The radioactive substance Carbon-14 ( $C^{14}$ ) has a half-life of about 5730 years. This means that if we measure the amount of  $C^{14}$  in a sample, then measure it again 5730 years later, we will find half as much  $C^{14}$  as we started with. A test tube contains 20 g of  $C^{14}$ . The tube is sealed and set aside for several thousand years, then the amount of  $C^{14}$  is measured again. How much  $C^{14}$  remains after 5730 years? After 11,460 years? After 17,190 years? After  $t$  years? What is the rate of change of the  $C^{14}$  at time  $t = 5730$ ?
- A metal plate is heated to a temperature of  $425^\circ\text{C}$  then allowed to cool in a room where the temperature is kept at  $25^\circ\text{C}$ . The plate cools so that the difference between the plate temperature and the room temperature decreases by half of the difference every  $1/4$  hour. What is the

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 temperature of the plate after 1/4 hour? After 1 hour? After  $t$  hours? What is the rate of change of the temperature at time  $t = 1$  hour?

**45.** Two thousand years ago Greek astronomers Hipparchus and Ptolemy created a system to quantify the apparent brightness of stars. The 20 brightest visible stars were denoted magnitude 1 stars, and the faintest visible stars were declared to be of magnitude 6. Other stars were assigned magnitudes based on their apparent brightness compared with those of magnitude 1 or 6. In 1850 Pogson refined the system. He noticed that magnitude 1 stars were about 100 times brighter than magnitude 6 stars. Since there are five steps of magnitude from 1 to 6, Pogson proposed that a star of magnitude  $m$  should be  $\sqrt[5]{100} \approx 2.512$  times brighter than a star of magnitude  $m + 1$ .

- Explain how Pogson came up with the number  $\sqrt[5]{100}$  and why this number is reasonable.
- Antares is a magnitude 1.2 star, and Marsik is a magnitude 5.3 star. Let  $b_1$  be the brightness of Antares and  $b_2$  be the brightness of Marsik. Find the numerical value of  $b_1/b_2$ .

**46.** Give a convincing argument that the equation  $e^{-2x} = k$  has a solution for every  $k > 0$ .

**47.** Give a convincing argument that the equation  $e^x + 3e^{3x} = k$  has a solution for every  $k > 0$ .

**48.** Consider the equation  $e^x + 3e^{-3x} = k$ . Find two positive  $k$  values for which this equation does *not* have a solution.

**49.** Suppose that  $b > 0$  and  $b \neq 1$ , and let  $r > 0$ . Give a convincing argument that the equation

$$b^x = r$$

has exactly one solution. Support your argument with graphs.

**50.** Let  $a$ ,  $b$ , and  $k$  be real numbers. For what values of  $a$  and  $b$  does the equation

$$e^{ax} + 2e^{bx} = k$$

have a solution for every  $k > 0$ ?

**51.** Compute  $e^{C_2}$ ,  $e^{C_3}$ , and  $e^{C_{10}}$ . Make a conjecture about the value of  $e^{C_b}$ .

- 52.**
- Differentiate both sides of the equation  $(ab)^x = a^x b^x$ , and use the result to show that  $C_a + C_b = C_{ab}$ .
  - Differentiate both sides of the equation  $(a/b)^x = a^x/b^x$ , and use the result to show that  $C_a - C_b = C_{a/b}$ .
  - Differentiate both sides of the equation  $(a^b)^x = a^{bx}$ , and use the result to show that  $bC_a = C_{a^b}$ .
  - What familiar function satisfies the identities proved in a, b, and c?

**53.** In finding the value of  $e$ , we looked for a number  $b$  for which

$$\frac{b^h - 1}{h} \approx 1$$

for small  $h$ . Solve this expression for  $b$  to obtain

$$b \approx (1 + h)^{1/h}.$$

Substitute small values of  $h$  in the expression for  $b$  and obtain an approximation to  $e$ . Argue that

$$\lim_{h \rightarrow 0} (1 + h)^{1/h} = e.$$

**54.** Find two different functions that have derivative  $e^x$ .

**55.** Find two different functions that have derivative  $e^{3x}$ . Check by finding the derivative of your answers.

**56. a.** Find the equation of the line that passes through the origin and is tangent to the graph of  $y = e^x$ .

**b.** Consider the equation

$$e^x = ax.$$

How many solutions does this equation have if  $a < 0$ ? If  $a = 0$ ? If  $a > 0$ ?

**57. a.** Investigate the behavior of  $x/e^x$ ,  $x^2/e^x$ ,  $x^{10}/e^x$ , and  $x^{50}/e^x$  for very large values of  $x$ . (You may do this numerically, graphically, or both.)

**b.** Make a conjecture about the behavior of  $x^n/e^x$  as  $x$  grows toward infinity.

**58.** Simple population models often describe population by an exponential formula

$$P(t) = P_0 e^{kt}$$

where  $P_0$  is the population at some starting time  $t = 0$  and  $k$  is some positive constant. For such populations, the rate of change is

$$P'(t) = kP_0 e^{kt}.$$

Write a short paragraph explaining the meaning of this rate of change. In particular, explain why larger populations increase faster than smaller ones.

**59.** If amount  $A_0$  of a radioactive material is present at time  $t = 0$ , then the amount  $A(t)$  present at a later time  $t > 0$  is given by

$$A(t) = A_0 e^{-kt}$$

where  $k$  is a positive constant. Find the rate of change of the amount of radioactive material with respect to time, and write a short paragraph explaining the meaning of this rate of change. When radioactive material decays, it gives off radiation. Use the rate-of-change equation to explain why it is more dangerous to be near a large amount of radioactive material than near a small amount.