Exercise 43 for an alternative argument using the definition of derivative and some of the limit results found in Chapter 1. Keep in mind that all angles are measured in radians. As we shall see later in this section, the calculus of trigonometric functions is easier if we use radians instead of degrees. Please refer to the Appendix if you need to review some basic trigonometry.

INVESTIGATION

The Derivative of the Sine Function

The derivative of the sine function at a value θ is the slope of the "line" we see as we zoom in on the graph of $y = \sin\theta$ near the point $(\theta, \sin\theta)$. Figure 2.16 shows $(\theta, \sin\theta)$ and a nearby point $(\theta + h, \sin(\theta + h))$. To find the derivative of the sine function at θ, we calculate the slope

$$\frac{\sin(\theta + h) - \sin\theta}{h} = \frac{\Delta y}{h} \tag{1}$$

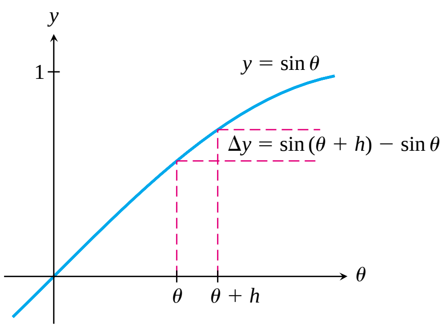

FIGURE 2.16 Finding the "slope" of the sine graph.

of the segment determined by these two points, then evaluate the limit of this slope as h approaches 0. In (1) we have set $\Delta y = \sin(\theta + h) - \sin\theta$ to denote the difference in the y-coordinates of the two points.

To calculate the limit of (1), we first approximate the numerator Δy. Consider the points $P = (\cos\theta, \sin\theta)$ and $Q = (\cos(\theta + h), \sin(\theta + h))$ on the unit circle, as shown in Fig. 2.17. Note that the vertical distance between P and Q is $\Delta y = \sin(\theta + h) - \sin\theta$. In the zoom-view of the line segment joining P and Q, we show a right triangle with one side Δy. We argue that the hypotenuse of this triangle is approximately h and the angle at Q is approximately θ.

Because h is small and angle POQ measures h radians,

$$\text{length of segment } PQ \approx \text{length of arc } PQ = h \cdot 1 = h. \tag{2}$$

For small h, segment PQ is very nearly tangent to the circle at P. Referring to the lower sketch in Fig. 2.17, it follows that angle OPQ is very nearly a right angle, and that the angle at Q is very nearly θ. Hence

$$\cos\theta \approx \frac{\Delta y}{h}. \tag{3}$$

Solving for Δy in this last equation, we see

$$\Delta y \approx h\cos\theta.$$

Substituting this result into (1) and letting $h \longrightarrow 0$, we have

$$\frac{d}{d\theta}\sin\theta = \lim_{h\to 0}\frac{\sin(\theta + h) - \sin\theta}{h} = \lim_{h\to 0}\frac{\Delta y}{h} = \lim_{h\to 0}\frac{h\cos\theta}{h} = \cos\theta.$$

Once we have the derivative of the sine function, the derivatives of the other trigonometric functions follow easily.