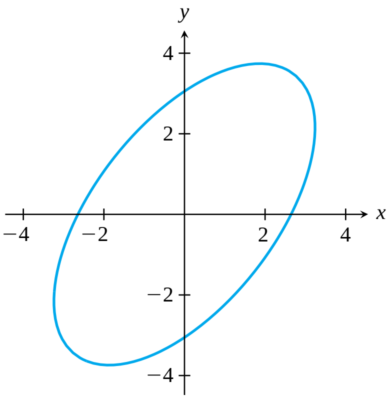

Figure for Exercise 31.

- **a.** Find the coordinates of all points where the ellipse crosses the coordinate axes.
- **b.** Find the coordinates of all points where the tangent line is horizontal.
- **c.** Find the coordinates of all points where the tangent line is vertical.

32. Consider the graph of all points (x, y) that satisfy the equation

$$2x^2 - 3xy + y^2 = -2.$$

- **a.** Find the coordinates of all points on the graph where the tangent line is horizontal.
- **b.** Find the coordinates of all points on the graph where the tangent line is vertical.
- **c.** Show that there are no points on the graph where the tangent line has slope 1.

2.5 Trigonometric Functions

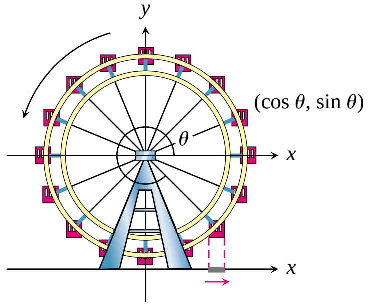

FIGURE 2.15 The shadow of a car on a Ferris wheel moves back and forth along the ground.

If you are ever at a carnival near noon on a sunny day, go look at the Ferris wheel. When the sun is directly overhead, the shadow of the Ferris wheel will be on the ground below the wheel. Imagine watching the shadow of one of the cars on the Ferris wheel. The shadow will move back and forth across the ground as the wheel turns. The moving shadow illustrates two very important ideas. First, the moving shadow is an example of *periodic motion*. The back-and-forth motion of the shadow repeats regularly: The shadow moves from left to right under the Ferris wheel, then from right to left, then from left to right, and so on. Periodic phenomena are everywhere in our world. They are seen in the tides, the orbit of Earth about the sun, the phases of the moon, and the motion of a piston in an internal combustion engine.

The moving shadow also illustrates the cosine function. Let the Ferris wheel have radius 1, and put a coordinate system in the plane of the wheel with the axle at the origin and the y-axis perpendicular to the ground. See Fig. 2.15. Let θ be the angle the ray from the origin through the car makes with the positive x-axis. The x-coordinate of the car is $\cos\theta$. Because the position of the car's shadow on the ground is just this x-coordinate, you are seeing an animation of the cosine function. This example illustrates the close relationship between periodic phenomena and trigonometric functions.

Now look more closely at the motion of the shadow. You will notice that the shadow moves very slowly near the ends of its back-and-forth motion, and it moves faster when it is under the center of the wheel. Because velocity is the rate of change of position with respect to time, the speed of the shadow is related to the derivative of the cosine function.

Java Applet
Transformations of Periodic Functions
Graphs periodic functions of the form $y = A\cos(Bx + C) + D$ and $y = A\sin(Bx + C) + D$, showing the results of changing the values of A, B, C, and D. Thus, students can explore affects of changes involving amplitude, phase shift, and period.

The Derivatives of Sine and Cosine

We use the geometry of the unit circle to find the derivative of the sine function. While the argument is not "mathematically rigorous," it does give some geometric insight into the relationship between the sine function and its derivative. See

Exercise 43 for an alternative argument using the definition of derivative and some of the limit results found in Chapter 1. Keep in mind that all angles are measured in radians. As we shall see later in this section, the calculus of trigonometric functions is easier if we use radians instead of degrees. Please refer to the Appendix if you need to review some basic trigonometry.

INVESTIGATION

The Derivative of the Sine Function

The derivative of the sine function at a value θ is the slope of the "line" we see as we zoom in on the graph of $y = \sin\theta$ near the point $(\theta, \sin\theta)$. Figure 2.16 shows $(\theta, \sin\theta)$ and a nearby point $(\theta + h, \sin(\theta + h))$. To find the derivative of the sine function at θ, we calculate the slope

$$\frac{\sin(\theta + h) - \sin\theta}{h} = \frac{\Delta y}{h} \tag{1}$$

of the segment determined by these two points, then evaluate the limit of this slope as h approaches 0. In (1) we have set $\Delta y = \sin(\theta + h) - \sin\theta$ to denote the difference in the y-coordinates of the two points.

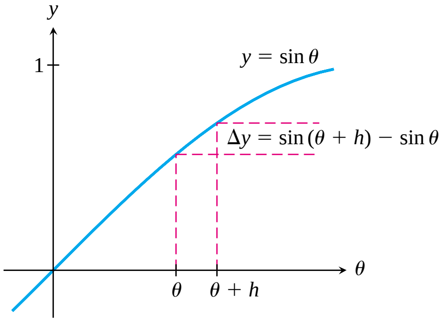

FIGURE 2.16 Finding the "slope" of the sine graph.

To calculate the limit of (1), we first approximate the numerator Δy. Consider the points $P = (\cos\theta, \sin\theta)$ and $Q = (\cos(\theta + h), \sin(\theta + h))$ on the unit circle, as shown in Fig. 2.17. Note that the vertical distance between P and Q is $\Delta y = \sin(\theta + h) - \sin\theta$. In the zoom-view of the line segment joining P and Q, we show a right triangle with one side Δy. We argue that the hypotenuse of this triangle is approximately h and the angle at Q is approximately θ.

Because h is small and angle POQ measures h radians,

$$\text{length of segment } PQ \approx \text{length of arc } PQ = h \cdot 1 = h. \tag{2}$$

For small h, segment PQ is very nearly tangent to the circle at P. Referring to the lower sketch in Fig. 2.17, it follows that angle OPQ is very nearly a right angle, and that the angle at Q is very nearly θ. Hence

$$\cos\theta \approx \frac{\Delta y}{h}. \tag{3}$$

Solving for Δy in this last equation, we see

$$\Delta y \approx h\cos\theta.$$

Substituting this result into (1) and letting $h \longrightarrow 0$, we have

$$\frac{d}{d\theta}\sin\theta = \lim_{h\to 0}\frac{\sin(\theta + h) - \sin\theta}{h} = \lim_{h\to 0}\frac{\Delta y}{h} = \lim_{h\to 0}\frac{h\cos\theta}{h} = \cos\theta.$$

Once we have the derivative of the sine function, the derivatives of the other trigonometric functions follow easily.

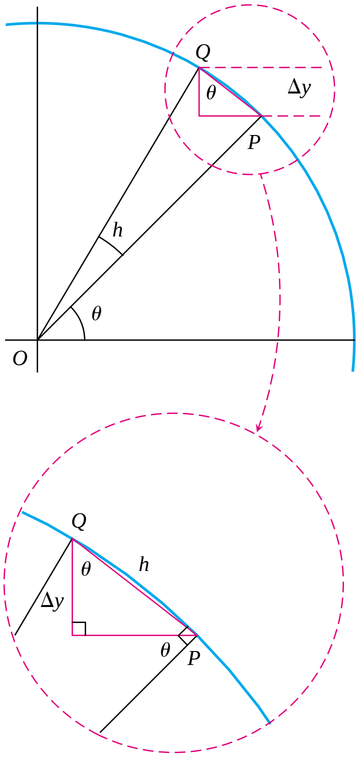

FIGURE 2.17 Finding Δy on the unit circle.

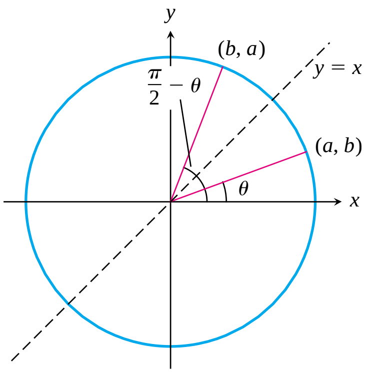

FIGURE 2.18 $\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin \theta$
$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos \theta$

EXAMPLE 1 Find $\dfrac{d}{dx} \cos x$.

Solution

Let $w = \cos x$ and recall that

$$w = \cos x = \sin\left(\frac{\pi}{2} - x\right).$$

See Fig. 2.18 for a pictorial proof of this identity, or see the review of trigonometry in the Appendix. This equation expresses $\cos x$ as a composition of the sine function and the function given by $\dfrac{\pi}{2} - x$. As a composition chain this is

$$w = \sin u$$

$$u = \frac{\pi}{2} - x.$$

By the chain rule,

$$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx} = (\cos u)(-1) = -\cos\left(\frac{\pi}{2} - x\right) = -\sin x.$$

Summarizing the last two results, we have:

Derivatives of the Sine and Cosine

$$\frac{d}{dx} \sin x = \cos x \quad \text{and} \quad \frac{d}{dx} \cos x = -\sin x \qquad (4)$$

EXAMPLE 2 Find $\dfrac{d}{dx} \tan x$.

Solution

Because

$$\tan x = \frac{\sin x}{\cos x}$$

we can use the quotient rule developed in Example 4 of Section 2.2. We have

$$\frac{d}{dx} \tan x = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

Any of the differentiation techniques studied in earlier sections may be needed to find the derivatives of the functions that involve trigonometric functions.

EXAMPLE 3 Find $\dfrac{dy}{dx}$ if $y = \sqrt{x \cos x}$.

Solution

First decompose the function into a composition chain:

$$y = \sqrt{u}$$
$$u = x \cos x$$

We then have

$$\frac{dy}{du} = \frac{d}{du}u^{1/2} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}},$$

and by the product rule

$$\frac{du}{dx} = (x)' \cos x + x(\cos x)' = 1 \cdot \cos x + x(-\sin x) = \cos x - x \sin x.$$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{2\sqrt{u}}\right)(\cos x - x \sin x).$$

Substituting $u = x \cos x$ in the last expression, we obtain the derivative,

$$\frac{dy}{dx} = \frac{\cos x - x \sin x}{2\sqrt{x \cos x}}.$$

EXAMPLE 4 Find the equation of the line tangent to the graph of

$$y = \cos(\pi x^2) \tag{5}$$

at the point $\left(1/2, 1/\sqrt{2}\right)$. Use the tangent line to approximate the value of $\cos(\pi(0.55)^2)$.

Solution

To find the slope of the tangent line, we first need to calculate dy/dx. A composition chain for (5) is

$$y = \cos u$$
$$u = \pi x^2.$$

By (4),

$$\frac{dy}{du} = -\sin u.$$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\sin u)(2\pi x) = -2\pi x \sin(\pi x^2).$$

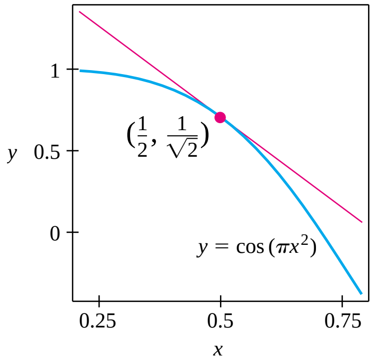

FIGURE 2.19 The line $y = \dfrac{1}{\sqrt{2}} - \dfrac{\pi}{\sqrt{2}}\left(x - \dfrac{1}{2}\right)$ is tangent to the graph of $y = \cos(\pi x^2)$ at the point $\left(1/2, 1/\sqrt{2}\right)$.

The slope of the line tangent to the graph at $\left(1/2, 1/\sqrt{2}\right)$ is the value of the derivative at this point:

$$\text{slope} = -2\pi(1/2)\sin(\pi(1/2)^2) = -\pi \sin\left(\frac{\pi}{4}\right) = -\frac{\pi}{\sqrt{2}}.$$

Because the tangent line has slope $-\pi/\sqrt{2}$ and contains the point $\left(1/2, 1/\sqrt{2}\right)$, an equation for the line is

$$y - \frac{1}{\sqrt{2}} = -\frac{\pi}{\sqrt{2}}\left(x - \frac{1}{2}\right),$$

or

$$y = \frac{1}{\sqrt{2}} - \frac{\pi}{\sqrt{2}}\left(x - \frac{1}{2}\right). \tag{6}$$

The graph of $y = \cos(\pi x^2)$ and the graph of the tangent line are shown in Fig. 2.19. Because 0.55 is relatively close to $1/2 = 0.5$, we may approximate $\cos(\pi(0.55)^2)$ with the y value of the tangent line equation at $x = 0.55$. Thus

$$\cos(\pi(0.55)^2) \approx \frac{1}{\sqrt{2}} - \frac{\pi}{\sqrt{2}}\left(0.55 - \frac{1}{2}\right) = \frac{1}{\sqrt{2}} - \frac{\pi}{\sqrt{2}}(0.05) \approx 0.596.$$

How does this compare with the calculator value of $\cos(\pi(0.55)^2)$?

EXAMPLE 5 The equation

$$x = \tan y \tag{7}$$

implicitly defines y as a function of x. Find a formula for $\dfrac{dy}{dx}$.

Solution

Display (7) as a composition chain:

$$\begin{aligned} x &= \tan y \\ y &= y(x), \end{aligned}$$

where we have written $y = y(x)$ to remind ourselves that y is a function of x. By the chain rule,

$$\frac{dx}{dx} = \frac{dx}{dy}\frac{dy}{dx}. \tag{8}$$

Because $\dfrac{dx}{dx} = 1$ and $\dfrac{dx}{dy} = \dfrac{d}{dy}\tan y = \sec^2 y$, equation (8) becomes

$$1 = \sec^2 y \frac{dy}{dx}.$$

Solving for $\dfrac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}. \tag{9}$$

There is nothing wrong with this answer, but we can put it in a form that might be easier to work with. Noting that

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2,$$

we see that (9) can be written as

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

Derivatives of the Other Trigonometric Functions

We present here, for convenience, the derivatives of the other trigonometric functions. See Exercise 36.

Derivatives of Tangent, Cotangent, Secant, and Cosecant

$$\frac{d}{d\theta}\tan\theta = \sec^2\theta \qquad \frac{d}{d\theta}\cot\theta = -\csc^2\theta$$

$$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta \qquad \frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta.$$

Why We Work in Radians

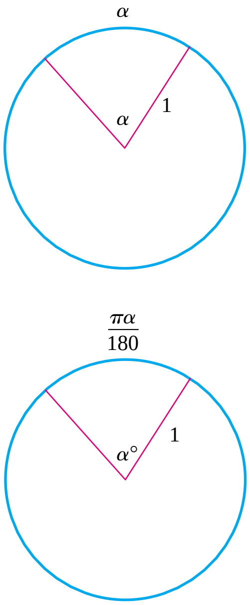

FIGURE 2.20 Arc length and the central angle

As mentioned earlier in this section, the calculus of trigonometric functions is easier to manage if angles are measured in radians. This is because of the relationship between central angle measure and arc length on the unit circle. A central angle of α radians subtends an arc of length α on the unit circle, while a central angle α *degrees* subtends an arc of length $\pi\alpha/180$. See Fig. 2.20. Now suppose that in the Investigation we had worked in degrees instead of radians. Then, with h measured in degrees, (2) would have been

$$\text{length of segment } PQ \approx \text{length of arc } PQ = \frac{\pi h}{180}. \tag{10}$$

Then (3) would have been

$$\cos\theta \approx \frac{\Delta y}{\pi h/180}, \tag{11}$$

leading to

$$\Delta y \approx h\frac{\pi}{180}\cos\theta.$$

Substituting this result into (1) would have resulted in

$$\frac{d}{d\theta}\sin\theta = \lim_{h\to 0}\frac{\sin(\theta+h)-\sin\theta}{h} = \lim_{h\to 0}\frac{\Delta y}{h} = \lim_{h\to 0}\frac{h\frac{\pi}{180}\cos\theta}{h} = \frac{\pi}{180}\cos\theta.$$

This result illustrates a reason for working in radians rather than degrees. When we use radian measure, the derivative of the sine function is "cleaner." In degrees, the derivative carries an inconvenient constant multiplier of $\pi/180$.