

31

Fourier Series

Tools Used in Lab 31

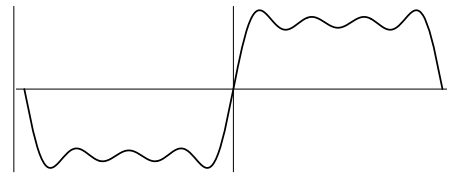
- Fourier Series: Square Wave
- Fourier Series: Triangle Wave
- Fourier Series: Gibbs Effect
- Fourier Series: Coefficients

How does a Fourier series converge to a function? What is the difference between a Fourier series and a Maclaurin series? Why is the convergence bad near a discontinuity of the function?

1. Introduction

In 1807, Joseph Fourier announced that an “arbitrary” function $f(t)$ can be represented in terms of sine and cosine functions:

$$f(t) = \sum_{n=0}^{\infty} a_n \cos nt + b_n \sin nt$$



In 1822, Fourier published many examples of such representations. Prior to Fourier, piecewise-defined functions like the square wave and triangle wave were not considered to be “real” functions. The effort to clarify and prove Fourier’s claims led to the modern idea of function (which is familiar to all calculus students).

2. Square Wave

Use the **Fourier Series: Square Wave** tool to answer the following questions on approximating a square wave with a Fourier series.

- 2.1 How does the approximation using five terms of the Fourier series differ from the approximation using four terms of the Fourier series? If you have trouble with this question, look at the approximation using one term, the approximation using two terms, and the approximation using three terms.
- 2.2 Is the approximation using five terms of the Fourier series better than the approximation using four terms at every point? Discuss.

- 2.3 In what sense is the approximation using five terms better than the approximation using four terms?
- 2.4 In general, if one uses n terms of the Fourier series, how many waves (relative maxima) will there be between 0 and π ? Relate your answer to the n^{th} term of the series.
- 2.5 Does the maximum error decrease as the number of terms in the Fourier series approximation increases? Discuss. (The phenomenon that occurs here is called the **Gibbs effect** and occurs near any discontinuity of a function being approximated by a Fourier series.)
- 2.6 Does the square wave have a Maclaurin series expansion? Discuss.
- 2.7 For what functions should one try to use Fourier series approximations and for what functions should one use Taylor series?

3. Triangle Wave

Use the **Fourier Series: Triangle Wave** tool to answer the following questions.

- 3.1 Is the approximation using five terms of the Fourier series better than the approximation using four terms at every point? Discuss.
- 3.2 In what sense is the approximation using five terms better than the approximation using four terms?
- 3.3 Does the maximum error decrease as the number of terms in the Fourier series approximation increases? Why does the Gibbs effect not occur? (See Section 4.)

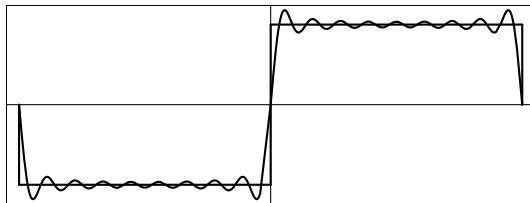
3.4 Does the triangle wave have a Maclaurin series expansion? Discuss.

4. Additional Exercises

4.1 By looking at the graphs of the square wave and the triangle wave, convince yourself that the square wave function is the derivative of the triangle wave function. Explain in words.

4.2 Verify that the Fourier series for the square wave can be obtained by differentiating the Fourier series for the triangle wave term by term.

5. Gibbs Effect



The Gibbs effect is the spike that occurs in a Fourier series approximation near a discontinuity of a function. Use the **Fourier Series: Gibbs Effect** tool and the fact that the square wave is the derivative of the triangle wave (see Exercises 4.1 and 4.2) to convince yourself that the Gibbs effect has to occur and to answer the following questions.

5.1 Does the maximum error in the Fourier series approximation of the square wave go to zero or do you think it remains larger than some positive value? If so, estimate that value.

5.2 How does the tip of the triangle wave (which occurs at $t = \pi$) manifest itself in the square wave and why?

- 5.3 By examining the Fourier series approximation of the triangle wave near π , explain why the Gibbs effect has to occur in the square wave.

6. Arbitrary Fourier Series

The **Fourier Series: Coefficients** tool can be used to explore how the coefficients in a Fourier series are related to properties of the function. Good functions to play with are square waves, triangle waves, and sawtooth waves of various periods. Imagine a function shape and try to choose the coefficients, using the sliders to approximate the functions.

Even functions have Fourier series involving only the constant term and the cosine terms. Odd functions have Fourier series involving only the sine terms. An arbitrary function can be expressed as the sum of an even and an odd function.

- 6.1 Show that if $f(t)$ is an arbitrary function then $g(t) = \frac{f(t) + f(-t)}{2}$ is even, $h(t) = \frac{f(t) - f(-t)}{2}$ is odd, and $f(t) = g(t) + h(t)$.
- 6.2 What distinguishes functions whose Fourier series involves only terms of the form $\sin nt$ with n even? with n odd?
- 6.3 What distinguishes functions whose Fourier series involves only terms of the form $\cos nt$ with n even? with n odd?

Lab 31: Tool Instructions

Fourier Series: Square Wave Tool

Buttons

- Click the mouse on a button in the upper group to view the graph of a partial series sum.
- Click the mouse on a button in the middle group to view overlays of partial series sums.
- Click the mouse on a button in the lower group to view graphs of individual series terms.

Fourier Series: Triangle Wave Tool

Buttons

- Click the mouse on a button in the upper group to view the graph of a partial series sum.
- Click the mouse on a button in the middle group to view overlays of partial series sums.
- Click the mouse on a button in the lower group to view graphs of individual series terms.

Fourier Series: Gibbs Effect Tool

Buttons

- Click the mouse on a button to view the graphs for a partial series sum.
- The upper graphs show the triangle wave, with magnification on the left to see Gibbs effect.
- The lower graphs show the square wave, derivative of the triangle wave, with the magnification on the left to see Gibbs effect.

Fourier Series: Coefficients Tool

Parameters Sliders

- Use the sliders to change the values for the parameters k and c .
- Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

- Click the mouse on the **[Cosine Series]** or **[Sine Series]** buttons to choose a series.
- Click the mouse on the lower set of buttons to select which terms to display: all, odd, or even.

