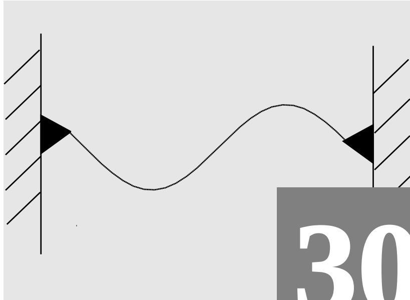

30 Boundary Value Problems

Tools Used in Lab 30

- Boundary Values: Eigenvalues
- Boundary Values: Solvability and Uniqueness
- Bessel Function: 1st Kind
- Bessel Equation

The vibration of a string, the motion of a drumhead are examples of physical systems whose behavior is determined by values of a function at the two endpoints of an interval rather than just at the initial point.

1. The Vibrating String

A mode of vibrating string satisfies the differential equation:

$$y'' + \lambda y = 0 \qquad (1)$$

The ends of the string are held fixed so $y(0) = y(1) = 0$. Values of λ for which a boundary value problem has a non-trivial solution are called **eigenvalues.**

1.1 Solve Equation (1) and use the solution to figure out the eigenvalues.

Open the **Boundary Values: Eigenvalues** tool to verify that your eigenvalues in Exercise **1.1** are correct and to see some solutions of Equation (1). Be sure to click on the button $y(1) = 0$. The technique you use with the **Boundary Values: Eigenvalues** tool to look for eigenvalues is called **shooting**; you try a value of λ and "shoot" for the boundary condition $y(1) = 0$. You adjust your aim until you hit the boundary condition. In case you are a poor shot, the tool provides a "find closest eigenvalue" button.

Generally, if the string is plucked or strummed or somehow caused to vibrate, the equation of motion of the string will be given by

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos(n\pi ct - \delta_n) \sin n\pi x$$

The constant c depends on the string, and the factors sin $n\pi x$ are the solutions to Exercise **1.1** satisfying the initial conditions $y(0) = y(1) = 0$, and are called the **fundamental modes** of vibration. Each one of the fundamental modes is multiplied by a term $a_n \cos(n\pi ct - \delta_n)$ to give the vibratory motion in that mode — a_n and δ_n are the amplitude and phase shift of the vibration in the n^{th} fundamental mode. The total motion of the string is the sum of the motions in the fundamental modes. How each fundamental mode contributes is determined by a_n and $\delta_{n,}$ which in turn are determined by the initial conditions, which will be different depending on how the string is caused to vibrate.

2. A Different Set of Boundary Conditions

Instead of the boundary conditions $y(0) = y(1) = 0$, one could take the boundary conditions $y(0) = y'(1) = 0$ for Equation (1).

2.1 Use the solution to Equation (1) from Exercise **1.1** to figure out the eigenvalues with this new boundary condition.

Use the **Boundary Values: Eigenvalues** tool to verify that your eigenvalues in Exercise **2.1** are correct and to see some solutions of Equation (1) with these new boundary conditions. Be sure to click on the button $y'(1) = 0$.

3. Solvability and Uniqueness

In this section we consider the equation:

$$y'' + ky = \sin 2x, k > 0 \qquad (2)$$

and look for solutions to the boundary value problem $y(0) = y(\pi) = 0$. Because of the sin2x on the right hand side, the values k for which there is a solution are not called eigenvalues, which is why we use k instead of λ. Also, the k in Equation (2) behaves differently from the λ in Equation (1). Open the **Boundary Values: Solvability and Uniqueness** tool. This tool allows you to “shoot” for the boundary condition $y(\pi) = 0$ both by varying k and by varying $y'(0)$. For the solutions of Equation (1) satisfying $y(0) = 0$, varying $y'(0)$ did not change whether or not $y(x)$ was a solution of the boundary value problem. Here the situation is different.

3.1 Take $k = 4$. Can you find a solution for the boundary value problem? What value(s) can $y(\pi)$ assume?

3.2 Take $k = 5$. Can you find a solution of the boundary value problem? Is the solution unique? That is does more than one value of $y'(0)$ give a solution?

3.3 Take $k = 9$. Can you find a solution of the boundary value problem? Is the solution unique? That is does more than one value of $y'(0)$ give a solution?

3.4 Solve Equation (2) with initial conditions $y(0) = 0$.

3.5 Use your answer to Exercise **3.4** to tell for what values of k a solution to the boundary value problem of Equation (2) with $y(0) = y(\pi) = 0$ will exist and for what values of k the solution will be unique. When the solution exists (whether or not it is unique) indicate what value(s) of $y'(0)$ give a solution. If the solution does not exist, indicate why.

Note: Equation (2) replaced by Equation (3) (that is, with the 2 on the right side changed to an arbitrary positive number a) provides an interesting variation on Exercises **3.1–3.5**.

$$y'' + ky = \sin ax, k > 0 \tag{3}$$

4. The Vibration of a Drumhead

Infinite series techniques can be used to solve Bessel's equation of order p:

$$t^2 x'' + tx' + \left(t^2 - p^2\right)x = 0, \quad 0 < t < \infty. \tag{4}$$

Frobenius's method gives a solution, denoted $J_p(t)$, called the Bessel function of the first kind of order p. Bessel's equation is singular at $t = 0$ but for $p \geq 0$, $J_p(t)$ is continuous, $t \geq 0$. Open the **Bessel Function: 1st Kind** tool to recall the behavior of $J_p(t)$.

4.1 Use the **Bessel Function: 1st Kind** tool to find the first few zeros $\lambda_1, \lambda_2, \ldots$ of $J_0(t)$.

4.2 Show that under the change of independent variable $t = \lambda r$, Equation (4) is changed into the equation $R'' + \frac{1}{r}R' + \left(\lambda^2 - \frac{p^2}{r^2}\right)R = 0$. That is, $x(t)$ is a solution of $t^2x'' + tx' + \left(t^2 - p^2\right)x = 0$ if and only if $R(r) = x(\lambda r)$ is a solution of $R'' + \frac{1}{r}R' + \left(\lambda^2 - \frac{p^2}{r^2}\right)R = 0$.

Consider the boundary value problem (this is just the equation of Exercise **4.2** with $p = 0$):

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0, R(0) = 1, R(1) = 0 \qquad (5)$$

4.3 Use the results of Exercises **4.1** and **4.2** to express the solutions of Equation (5) in terms of $J_0(t)$ and to find the first few eigenvalues of Equation (5) (values of λ for which Equation (5) has a solution).

Open the **Bessel Equation** tool to verify your answer in Exercise **4.3**. Again, you use the shooting method with this tool to find eigenvalues.

The motion of a circular drumhead of radius 1 struck exactly in the center is given by

$$H(t,r) = \sum_{k=1}^{\infty} a_k \cos\left(\lambda_k \omega t - \delta_k\right) J_0\left(\lambda_k r\right)$$

where ω is the natural frequency of the drum, the λ_k are the zeros of the Bessel function of the first kind of order zero, r is radial distance of a point from the center of the drumhead, a_k is amplitude, and δ_k is the phase shift of $J_0(\lambda_k r)$'s contribution to the vibration, and $H(t,r)$ is the height of a point at distance r from the center of the drumhead above the rest position at time t.

The Bessel functions $J_p(t)$ of the first kind of order p (with p an integer) are similarly related to vibrations of a drumhead with more complicated initial conditions.

Lab 30: Tool Instructions

Boundary Values: Eigenvalues Tool

Parameter Sliders

Use the slider to set the value for the parameter λ.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Buttons

Click the mouse on the **[Clear]** button to clear all the trajectories from the graph.
Click the mouse on the **[Clear to last]** button to clear all trajectories except the last one drawn.
Click the mouse on the **[Find nearest eigenvalues]** button to find the nearest eigenvalues.
Click the mouse on the **[y(1)=0]** or **[y'(1)=0]** button to choose the right boundary.

Boundary Values: Solvability and Uniqueness Tool

Parameter Sliders

Use the slider to set the value for k and $y'(0)$.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Buttons

Click the mouse on the **[Clear]** button to clear all the trajectories from the graph.
Click the mouse on the **[Clear to last]** button to clear all trajectories except the last one drawn.
Click the mouse on the **[Find nearest solution]** button to locate the nearest solution.

Bessel's Function: 1st Kind

Parameter Slider

Use the slider to set p, the order of the function.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Bessel's Equation Tool

Parameter Sliders

Use the slider to set the value for the parameter λ.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Buttons

Click the mouse on the **[Clear]** button to clear all the trajectories from the graph.
Click the mouse on the **[Clear to last]** button to clear all trajectories except the last one drawn.
Click the mouse on the **[Find nearest eigenvalues]** button to find the nearest eigenvalues.