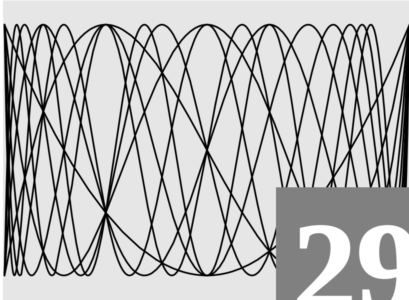

29 Special Functions (from Series Solutions)

Tools Used in Lab 29

- Chebyshev's Equation
- Lissajous Figures
- Bessel Function: 1st Kind

Power series solutions of differential equations give rise to new functions, functions that cannot be expressed in terms of sines, cosines, exponentials, logarithms, roots, or any familiar functions. How do these new functions behave and what are their properties?

1. Chebyshev's Equation

For $n = 0, 1, 2, \ldots$ Chebyshev's equation, $(1-t^2)x'' - tx' + n^2x = 0$, $-1 < t < 1$, has polynomial solutions. These polynomials are multiples of the **Chebyshev polynomials** $T_n(t) = \cos(n \arccos(t))$. The n^{th} Chebyshev polynomial $T_n(t)$ can be obtained by expressing $\cos n\theta$ in terms of $\cos \theta$ and substituting $t = \cos \theta$.

The formula $T_n(t) = \cos(n \arccos(t))$ makes sense even when n is not an integer and in fact, for any real number n, $T_n(t)$ is a solution of Chebyshev's equation. The function $S_n(t) = \sin(n \arccos(t))$ is also a solution of the Chebyshev equation, which is independent of $T_n(t)$ if $n > 0$.

Use the **Chebyshev's Equation** tool to answer the following questions.

1.1 Examine the graphs of the first few Chebyshev polynomials and give some properties of $T_n(t)$.

1.2 What is the value of: $T_n(0)$? $T'_n(0)$? $T_n(-1)$?

1.3 Give the formula for $T_{1/2}(t)$. What standard trig identity is this formula a restatement of? *(Hint:* Let $t = \cos(\theta)$.)

1.4 Give a formula for $T_2(t)$. What standard trig identity is this formula a restatement of?

2. DeMoivre's Formula: $\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$

2.1 Use DeMoivre's formula and the trig identity $\cos^2 \theta + \sin^2 \theta = 1$ to find formulas for the first few Chebyshev polynomials and verify that these are the polynomials given by the **Chebyshev's Equation** tool.

2.2 Compute the first few polynomials given by the recurrence relation, $T_0(t) = 1$, $T_1(t) = t$, $T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t)$, $n \geq 2$, and verify that these are the polynomials given by the **Chebyshev's Equation** tool.

2.3 Use infinite series methods to give a recurrence relation for the coefficients of the Maclaurin series expansion of the solutions to Chebyshev's equation with n arbitrary. For the first few non-negative integers, verify that Chebyshev's equation has a polynomial solution that is a multiple of the Chebyshev polynomial.

2.4 By differentiating $T_n(t) = \cos(n \arccos(t))$ twice, show that $T_n(t)$ is a solution of Chebyshev's equation for all real numbers n. Notice that your computations just as easily show that $x(t) = R \cos(\arccos(t) - \delta)$ is a solution for R and δ arbitrary.

2.5 Use DeMoivre's formula and the trig identity $\cos^2\theta + \sin^2\theta = 1$ to find formulas for the first few of the functions $S_n(t)$ and verify that these are the polynomials given by the **Chebyshev's Equation** tool.

2.6 Show that $T_n(t)$ and $S_n(t)$ are independent for $n > 0$.

2.7 Show that $c_1 \cos\theta + c_2 \sin\theta = R\cos(\theta - \delta)$ where $R = \sqrt{c_1^2 + c_2^2}$ and $\tan\delta = c_2 / c_1$ (thus $c_1 = R\cos\theta$ and $c_2 = R\sin\theta$). Conclude that the formula given on the **Chebyshev's Equation** tool is the correct formula for the general solution of Chebyshev's equation for $n > 0$.

2.8 When $n = 0$, $x(t)$ identically constant is a solution of Chebyshev's equation. Find a second independent solution of Chebyshev's equation when $n = 0$.

3. Lissajous Figures

Jules Antoine Lissajous (1822–1880) was interested in the physics of wave motion. Lissajous obtained his figures, in the context of acoustics, from the superposition of the vibrations of tuning forks, vibrating about mutually perpendicular axes. If Lissajous knew the characterisics (frequency and phase shift) of one tuning fork, he was able to deduce the characteristics of the other tuning fork from the **Lissajous figure**.

Lissajous figures are graphs in the xy-plane given parametrically by

$$x(t) = \cos(2m\pi t - a)$$
$$y(t) = \cos(2n\pi t - b)$$

We will see that Lissajous figures are related to Chebyshev polynomials.

For the following questions, use the **Lissajous Figures** tool.

3.1 If $a = 0$, $b = \pi/2$, $m = n = 1$, what is the Lissajous figure? Why?

3.2 If $m = n = 1$, how can one detect whether $x(t)$ and $y(t)$ are in phase, that is, that $a - b = 2k\pi$?

3.3 If $m = 1$ and $a = b = 0$, what is the Lissajous figure? Why?

3.4 How can we detect the ratio of the frequency of the wave $x(t)$ to the frequency of the wave $y(t)$, that is, the ratio m/n?

3.5 What will the Lissajous figure look like if m/n is not an integer?

4. Other Equations with Maclaurin Series Solutions

There are other interesting families of equations that have important applications, which have series solutions that are generally not expressible in terms of the elementary functions, but which have polynomial solutions for certain values of the parameter. Among such families of equations are

Hermite's Equation: $x'' - 2tx + 2nx = 0$;

Laguerre's Equation: $tx'' + (1 - t)x' + nx = 0, \quad t > 0$;

Legendre's Equation: $(1 - t^2)x'' - 2tx' + n^2x = 0, \quad -1 < x < 1$.

You can use a differential equation graphical solver to explore and analyze one of these.

5. Bessel's Equation

Bessel's equation of order p, $t^2x'' + tx' + (t^2 - p^2)x = 0$, $0 < t < \infty$, is singular at $t = 0$ and thus does not generally have Maclaurin series solutions. However, Frobenius's method gives a series solution, denoted $J_p(t)$, called the Bessel function of first kind of order p. The **Bessel Function: 1st Kind** tool graphs $J_p(t)$.

Lab 29: Tool Instructions

Chebyshev's Equation Tool

Parameter Slider

Use the slider to set the parameter n.
Chebyshev polynomials are found at integer values of n.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the **[Chebyshev Function]** button to get three parameters, r, n, s, and the buttons **[Chebyshev Cosine]** and **[Chebyshev Sine]**.

Lissajous Figures Tool

Parameter Slider

Use the slider to change the values for the parameter a, m, b, and n.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Bessel's Function: 1st Kind Tool

Parameter Slider

Use the slider to set p, the order of the function.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.