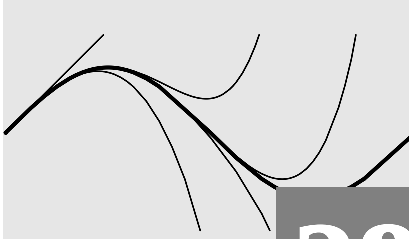

28 Maclaurin Series, Airy's Series

Tools in Used in Lab 28

- Maclaurin Series: Sine
- Maclaurin Series: Cosine
- Maclaurin Series: e^t
- Maclaurin Series: $1/\sqrt{1+t}$
- Airy's Equation
- Airy's Series: Cosine
- Airy's Series: Sine

How are functions approximated by their Maclaurin polynomials? How do the new functions that arise as solutions of the simple differential equation x" + tx = 0 behave and how are they similar to familiar functions?

1. Power Series for Elementary Functions

Recall that a Maclaurin series is a Taylor series about $t_0 = 0$. The n^{th} **Maclaurin polynomial** $p_n(t)$ (the sum of the terms of the Maclaurin series up to and including those of degree n) of a function $f(t)$ is the polynomial of degree n that, in some sense, best approximates the function at $t_0 = 0$.

$$p_n(t) = \sum_{k=0}^{n} f^{(k)}(0)\frac{t^k}{k!} \qquad (1)$$

The accuracy of the function can be judged by comparing it to the original function; the difference is the error. The error after n terms $E_n(t) = |f(t) - p_n(t)|$ is of **degree** $m > n$ in the sense that the error behaves like ct^m for t near 0. The n^{th} Maclaurin polynomial is the unique polynomial of degree n that approximates $f(t)$ with error of degree $m > n$. In fact, sufficiently near 0, the error is very well approximated by the absolute value of the first deleted term of the series.

Use the **Maclaurin Series: Sine**, the **Maclaurin Series: Cosine**, the **Maclaurin Series: e^t** and the **Maclaurin Series:** $1/\sqrt{1+t}$ tools to investigate how these series converge to the function. Select different numbers of terms and see how close the series approximations are. Investigate the fact that, near 0, the error is well approximated by the absolute value of the first deleted term. Try different values of m to see how individual terms of the series, shown in orange, approximate the error shown on the lower graph.

1.1 For the following functions $f(t)$ and integers n, give the degree m of the error of the approximation of $f(t)$ by the n^{th} Maclaurin polynomial:

a. $f(t) = \sin(t)$ $n = 3$; $m =$

b. $f(t) = \cos(t)$ $n = 2$; $m =$

c. $f(t) = e^t$ $n = 2$; $m =$

d. $f(t) =$ $f(t) = 1/\sqrt{1+t}$ $n = 2$; $m =$

1.2 In the **Maclaurin Series: Sine** and **Maclaurin Series: Cosine** tools, the absolute value of the first deleted term gives a pretty good approximation of the error in the range selected. How good do you think the approximation would be further from 0? Why?

1.3 In the **Maclaurin Series: *e*^{t}** tool, the absolute value of the first deleted term does not appear to give as good an approximation to the error as in the **Maclaurin Series: Sine** and **Maclaurin Series: Cosine** tools. Why is this?

1.4 In the Maclaurin series for $1/\sqrt{1+t}$, the absolute value of the first deleted term does not appear to give as good an approximation to the error as for the other series. Why is this?

1.5 Show that for $f(t) = e^t$ and $n = 2$, the error is well approximated near zero by the absolute value of the first deleted term; that is, show that $\left|e^t - \left(1+t+t^2/2!\right)\right|$ is well approximated by $|t|^3/3!$ near zero.

2. Additional Exercises

2.1 Use infinite series techniques to show that $x(t) = \sin(t)$ is a solution of the initial-value problem $x'' + x = 0,\ x(0) = 0,\ x'(0) = 1$

2.2 Use infinite series techniques to show that $x(t) = \cos(t)$ is a solution of the initial-value problem $x'' + x = 0,\ x(0) = 1,\ x'(0) = 0$.

3. Airy's Equation

Sir George Biddell Airy (1801–1892) was Astronomer Royal of England for many years and thus was interested in optics. Airy's equation, $x'' + tx = 0$ is used in studying the diffraction of light.

3.1 Open the **Airy's Equation** tool to see what the solutions look like. Sketch a phase plane solution in the xx'-plane.

3.2 Use the **Airy's Series: Cosine** tool to explore how the Maclaurin polynomials for the solution of Airy's equation $x'' + tx = 0$, with initial conditions $x(0) = 1$, $x'(0) = 0$, approximate the solution. What is the degree m of the error in the approximation of the Airy cosine function by the third Maclaurin polynomial? Why does the absolute value of the first deleted term appear to give so good an approximation to the error?

3.3 Use the **Airy's Series: Sine** tool to explore how the Maclaurin polynomials for the solution of $x'' + tx = 0$, with initial conditions $x(0) = 0$, $x'(0) = 1$, approximate the solution. What is the degree m of the error in the approximation of the Airy sine function by the first Maclaurin polynomial?

The differential equation $x'' + x = 0$ models a spring (without damping) with the spring constant equal to 1. Airy's equation $x'' + tx = 0$ can be thought of as modeling a spring with a variable spring "constant" t; that is, an aging spring that gets stiffer with time. Use the **Airy's Equation** tool to answer the following questions.

3.4 Compare and contrast the solutions $x(t)$ of these two equations satisfying the initial condition $x(0) = 1$, $x'(0) = 0$.

3.5 Compare and contrast the solutions $x(t)$ of these two equations satisfying the initial condition $x(0) = 0$, $x'(0) = 1$.

3.6 With initial conditions $x(0) = 1$, $x'(0) = 0$, the integral curve in the phase plane for $x'' + x = 0$ is a circle and for Airy's equation is something different. For $t > 0$, describe the difference and explain what differences in the graph of the corresponding solution curves $x(t)$ help explain that difference.

3.7 In general, at what points of the *tx*-plane will a solution of Airy's equation have a point of inflection?

3.8 Describe the behavior of the general solution $x(t)$ of Airy's equation when $t > 0$.

3.9 Describe the behavior of the general solution $x(t)$ of Airy's equation when $t < 0$.

3.10 The graph of the Airy cosine has a cusp in the phase plane. Let the lower graph finish plotting to see the cusp appear. Does the cusp indicate a lack of differentiability on the part of the Airy Cosine?

4. Additional Exercises

4.1 Use infinite series techniques to give two independent Maclaurin series solutions $x_1(t)$, $x_2(t)$ of the Airy's equation satisfying the initial conditions $x_1(0) = 1$, $x'_1(0) = 0$ and $x_2(0) = 0$, $x'_2(0) = 1$.

4.2 Use the substitution $u(t) = x'(t)/x(t)$ to transform the Airy equation into the equation $u'(t) = -u^2(t) - t$.

4.3 Some books give Airy's equation in the form $x'' - tx = 0$. Show that this equation is obtained from the equation $x'' + tx = 0$ by the change of coordinates $(x, t) \rightarrow (-x, -t)$. What effect does this change of coordinates have on solutions in the *tx*-plane?

Note: For a detailed analysis of the solutions to Airy's equation, see Hubbard, McDill, Noonburg, and West, *College Mathematics Journal* 25 (1994): 419–43.

Lab 28: Tool Instructions

Maclaurin Series: Sine Tool

Partial Series Buttons

Click a term button to select the number of terms for the partial sum.
Click the **[All]** button to overlay the first five approximations.

Parameter Slider

Use the slider to set the value of *m* in the error term.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Maclaurin Series: Cosine Tool

Partial Series Buttons

Click a term button to select the number of terms for the partial sum.
Click the **[All]** button to overlay the first five approximations.

Parameter Slider

Use the slider to set the value of *m* in error term.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Maclaurin Series: e^t Tool

Partial Series Buttons

Click a term button to select the number of terms for the partial sum.
Click the **[All]** button to overlay the first five approximations.

Parameter Slider

Use the slider to set the value of *m* in the error term.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Maclaurin Series: $1/\sqrt{1+t}$ Tool

Partial Series Buttons

Click a term button to select the number of terms for the partial sum.
Click the **[All]** button to overlay the first five approximations.

Parameter Slider

Use the slider to set the value of *m* in the error term.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Airy's Equation Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial conditions for a trajectory or a point for a vector.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Buttons

Click the mouse on the **[Cosine]** button to set the initial conditions for Airy's Cosine: $x(0) = 1$, $v(0) = 0$.
Click the mouse on the **[Sine]** button to set the initial conditions for Airy's Sine: $x(0) = 0$, $v(0) = 1$.
Click the mouse on the **[Clear]** button to remove all trajectories from the graph.

Airy's Series: Cosine Tool

Partial Series Buttons

Click a term button to select the number of terms for the partial sum.
Click the **[All]** button to overlay the first five approximations.

Parameter Slider

Use the slider to set the value of *m* in the error term.
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