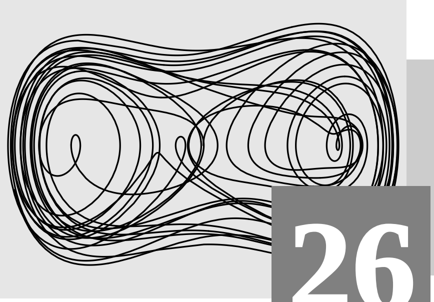


26 Chaos in Forced Nonlinear Oscillators

Tools Used in Lab 26

- Forced Damped Pendulum
- Forced Damped Pendulum: Poincaré Section
- Forced Damped Pendulum: Nine Sections
- Duffing Oscillator

What happens to the behavior of solutions as $t \to \infty$ in non autonomous, nonlinear, planar differential equations that model some simple applications?

1. Forced Damped Nonlinear Pendulum

The differential equation for the forced, damped, nonlinear pendulum is

$$\ddot{\theta} = -\sin\theta - b\dot{\theta} + A\cos\omega t.$$

Here A is the magnitude of the forcing function, ω is the frequency of the forcing function, and b measures damping. Letting $x = \theta$ and $y = \dot{\theta}$, we can write this as a system of two first order equations:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\sin x - by + A\cos\omega t.$$

Open the **Forced Damped Pendulum** tool.

1.1 Why isn't there a vector field drawn in the phase plane?

1.2 What difference is there in the solution curves of a planar autonomous system and a planar nonautonomous system?

1.3 With the default parameters ($A = 1.50$, $b = .50$, $\omega = 0.67$), how do the pendulum and corresponding solution curves behave as $t \to \infty$?

1.4 Change *A* to 1.35 and observe that the solution curves approach periodic behavior. Sketch the periodic orbit.

1.5 Change *A* to 1.45. What happens to the periodic orbit?

1.6 Change *A* to 1.47. What happens to the periodic orbit?

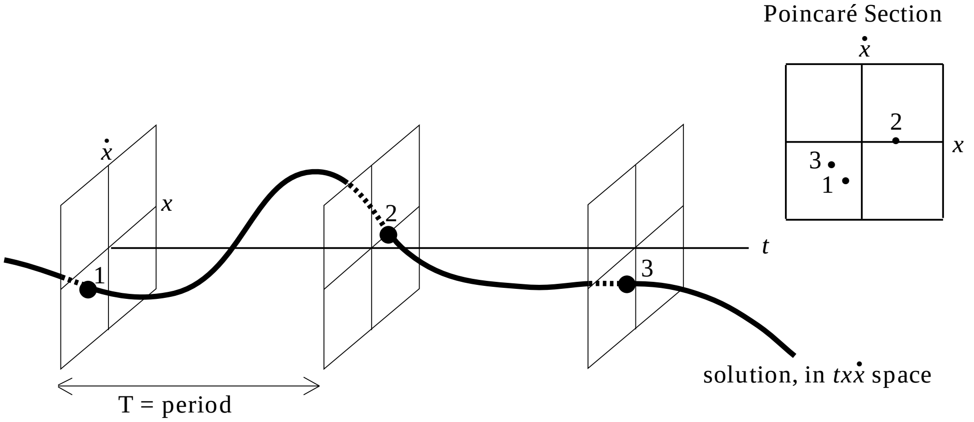
Systems that exhibit chaotic behavior often have values of the parameters where the solutions exhibit periodic behavior, as in Exercise **1.4**. Often as one changes the period slightly it will double, as in Exercise **1.5** and Exercise **1.6**, leading to the chaotic behavior of Exercise **1.3**.

1.7 If you wanted to make the origin an attractor, what strategy might you use for the parameters?

1.8 Do you think the sort of chaotic behavior of the solution curves as $t \rightarrow +\infty$ exhibited by the forced nonlinear pendulum could occur in a planar autonomous system? What are the limitations on the behavior a solution curve can exhibit as $t \rightarrow +\infty$ in a planar autonomous system?

2. Poincaré Section

The orbits of a nonautonomous system of two equations are a mess on a phase plane, because solutions can cross. It takes a three-dimensional phase space with axes for—typically—position, velocity, and time to sort solutions out to an uncrossed state. When we look at a planar representation, we are seeing a three-dimensional trajectory projected onto a plane. Though solutions for most systems do not cross in phase space, they often appear to cross when projected onto a plane. A loop floating in phase space that looks like an ellipse from above might look like an infinity symbol from the front.



The orbits in the phase portrait consist of all points on a solution curve. To construct a Poincaré section, we do not plot every point on an orbit, only points on planes, parallel to the xx'-plane at evenly spaced time intervals in the phase space. In the case of the **Forced Damped Pendulum: Poincare Section** and **Forced Damped Pendulum: Nine Sections** tools, the time interval is $T = 2\pi/\omega$. Starting with the point (x_1, y_1) in the plane $t = 0$, the trajectory intersects the plane $t = T$ at the point (x_2, y_2) and the plane $t = 2T$ at the point (x_3, y_3). We generate a sequence of points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$. These are the points plotted in the Poincaré section (they are labeled 1,2,3,…in the figure). Use the **Forced Damped Pendulum: Poincare Section** tool to get an idea of the concept of the Poincaré section.

The trajectory meeting the plane $t = T$ at (x_2, y_2, T) meets the plane $t = 2T$ at the point $(x_3, y_3, 2T)$. Because the forcing function is periodic with period T, the trajectory starting at $(x_2, y_2, 0)$ meets the plane $t = T$ at the point (x_3, y_3, T), and the trajectory from $(x_2, y_2, 0)$ to (x_3, y_3, T) is identical to the trajectory from (x_2, y_2, T) to $(x_3, y_3, 2T)$ translated left by T. Thus, in describing the dynamics of the forced pendulum, we can restrict our attention to the part of xyt-space between the planes $t = 0$ and $t = T$ — when a trajectory leaves this part of space through the plane $t = T$ at the point (x, y, T), it is considered to reenter this part of space through the plane $t = 0$ at the point $(x, y, 0)$. The Poincaré section is a slice through the figure consisting of the set of trajectories generated this way.

2.1 If a trajectory is periodic with period $3T$, how will it meet the part of space between the planes $t = 0$ and $t = T$?

2.2 How does the trajectory in Exercise **1.3** meet the part of space between the planes $t = 0$ and $t = T$?

The **Forced Damped Pendulum: Nine Sections** tool displays nine Poincaré sections taken at $\pi/(4\omega)$ time intervals in phase space. Use this tool to explore the next two questions (as usual, the **[Clear Transients]** button is useful in showing the attractor).

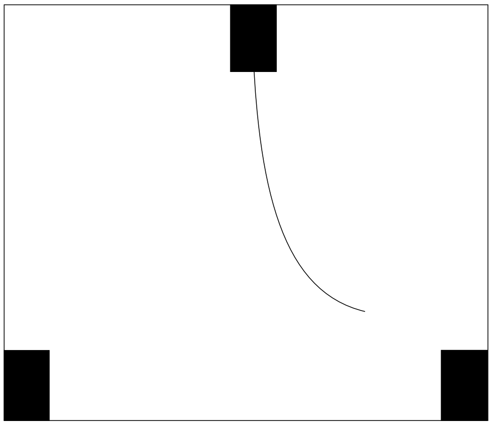
2.3 Will the Poincaré section depend on the sampling time? Discuss.

2.4 Why is the first picture the same as the last picture?

2.5 What should the Poincaré section look like if there is an attracting periodic orbit? *Hint:* Experiment with the parameter values that gave periodic orbits above. Use the **Forced Damped Pendulum: Poincare Section** tool.

3. Duffing's Equation

Consider a mechanical system in which a flexible steel beam is attached to the top, center of a box. There are magnets on either side of the box to which the beam is equally attracted. Duffing's equation describes the motion of the tip of the beam as the box is moved from side to side. The differential equation for the Duffing oscillator is:



$$\ddot{x} = -x\left(x^2 - k\right) - b\dot{x} + A\cos\omega t.$$

Here, A is the magnitude of the forcing function, ω is the frequency of the forcing function, b is the damping, and k is related to the stiffness of the beam and the strength of the magnets. The center of the box is at $x = 0$. Letting $y = \dot{x}$ we can write this equation as a system of two first-order equations:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x\left(x^2 - k\right) - by + A\cos\omega t.$$

Open the **Duffing Oscillator** tool.

3.1 With the default parameters ($k = 1$, $b = .15$, $A = .30$, $\omega = 1.00$), how do the tip of the steel beam and the corresponding solution curves behave as $t \longrightarrow +\infty$? Note that there are two completely different types of behavior depending on the choice of initial conditions. Describe both.

3.2 Take parameters $k = 1$, $b = .20$, $A = .30$, and $\omega = .60$ and sketch the resulting periodic orbit.

3.3 If you change b to .15, how does the periodic orbit change?

3.4 If you change b to .14, how does the periodic orbit change?

You are invited to explore other parameter values for the **Forced Damped Pendulum** and **Duffing Oscillator** tools.

Lab 26: Tool Instructions

Forced Damped Pendulum Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial conditions for a trajectory.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to set the damping constant *b*, the forcing amplitude *A*, and the forcing frequency *w*. Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled
☐ **position**
☐ **velocity**
☐ **acceleration**
toggle the time series on and off.

Other Buttons

Click the mouse on the **[Clear]** button to remove all the trajectories from the graphs.
Click the mouse on the **[Clear Transients]** button to remove all transient data from the graph without disrupting the active trajectory.
Click the mouse on the **[Pause]** button to stop a trajectory without canceling it.
Click the mouse on the **[Continue]** button to resume the motion of the paused trajectory

Force Damped Pendulum: Poincaré Section Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial conditions for a trajectory.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to set the damping constant *b*, the forcing amplitude *A*, and the forcing frequency *w*. Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled
☐ **position**
☐ **velocity**
toggle the time series on and off.

Other Buttons

Click the mouse on the **[Clear]** button to remove all the trajectories from the graphs.
Click the mouse on the **[Clear Transients]** button to remove all transient data from the graph without disrupting the active trajectory.
Click the mouse on the **[Pause]** button to stop a trajectory without canceling it.
Click the mouse on the **[Continue]** button to resume the motion of the paused trajectory

Forced Damped Pendulum: Nine Sections Tool

Setting Initial Conditions

Click the mouse on any of the nine graphing planes to set the initial conditions for a trajectory.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to set the damping constant *b*, the forcing amplitude *A*, and the forcing frequency *w*.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Other Buttons

Click the mouse on the **[Clear]** button to remove all the trajectories from the graphs.
Click the mouse on the **[Clear Transients]** button to remove all transient data from the graph without disrupting the active trajectory.

Duffing Oscillator Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial conditions for a trajectory.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to set the damping constant *b*, the forcing amplitude *A*, and the forcing frequency *w*, and the ratio of mechanical elastic force and electromagnetic force *k*.

Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled
[ ] **position**
[ ] **velocity**
[ ] **acceleration**
toggle the time series on and off.

Other Buttons

Click the mouse on the **[Clear]** button to remove all the trajectories from the graphs.
Click the mouse on the **[Clear Transients]** button to remove all transient data from the graph without disrupting the active trajectory.
Click the mouse on the **[Pause]** button to stop a trajectory without canceling it.
Click the mouse on the **[Continue]** button to resume the motion of the paused trajectory