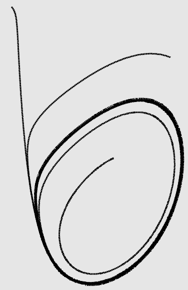


25 Bifurcations in Planar Systems

Tools Used in Lab 25
Chemical Oscillator
2-D Saddle-Node Bifurcation

Families of systems of first order autonomous equations, depending on a parameter, show surprising changes in behavior as the parameter is varied.

1. The Hopf Bifurcation

Lengyel, Rabai, and Epstein (*Journal of the American Chemical Society* (1990) 112, 9104) have posed and analyzed a particularly elegant model of an oscillating chemical reaction—the chlorine dioxide-iodine-malonic acid reaction—modeled by a system of two differential equations:

$$\begin{aligned}\frac{dx}{dt} &= a - x - \frac{4xy}{1+x^2}\\ \frac{dy}{dt} &= bx\left(1 - \frac{y}{1+x^2}\right)\end{aligned} \tag{1}$$

The variables x and y should be thought of as modeling concentrations of I^- and ClO_2^-, while a and b should be thought of as chemical constants. Also see Steven H. Strogatz, *Nonlinear Dynamics and Chaos*. Addison-Wesley, Reading, MA: 1994.

As the concentrations of I^- and ClO_2^-, change, the color of the solution changes. Open the **Chemical Oscillator** tool for a simulation of the chemical reaction and for help answering the following questions. If you have a friend who is a chemist you might enjoy mixing up a batch and seeing whether the color changes indicated in the simulation are accurate.

For simplicity, we take $a = 10$ and let $b > 0$ be a parameter.

1.1 Find all equilibria of Equation (1).

1.2 Linearize Equation (1) at the equilibrium and find the eigenvalues.

1.3 Conclude that the equilibrium is a repellor if $b < b_c = 3.5$, and an attractor if $b > b_c$.

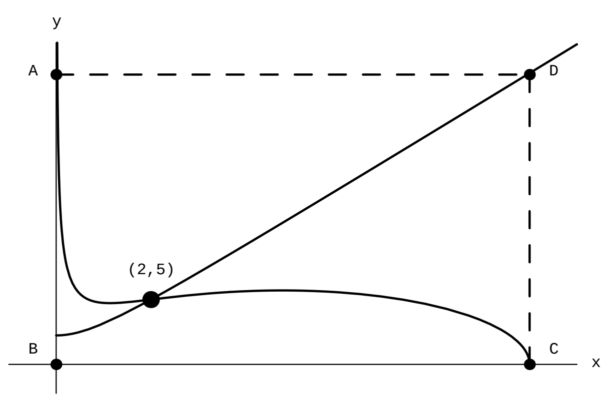
Let $A = (0,101)$, $B = (0,0)$, $C = (10,0)$, and $D = (10,101)$.

1.4 Show that the vector field points horizontally to the right along segment AB.

1.5 Show that the vector field points up along segment BC.

1.6 Show that the vector field points to the left along segment CD.

1.7 Show that the vector field points down along segment DA.



Let $\mathfrak{R}$ be the region bounded by the rectangle $ABCD$, with the point (2,5) deleted. By **1.4**, **1.5**, **1.6**, and **1.7** no trajectory leaves the region bounded by the rectangle. Further, if $b < b_c$, the point (2,5) is a repellor, so no trajectory can leave the region $\Re$. By the Poincaré-Bendixson theorem, the region $\Re$ contains a cycle. Use the **Chemical Oscillator** tool to convince yourself that this cycle is an attracting cycle that shrinks continuously to the equilibrium as b increases to b_c.

Equation (1) is said to have a **Hopf bifurcation** at b_c. That is, for $b > b_c$, (2,5) is an attracting equilibrium. At $b = b_c$ a bifurcation occurs. The equilibrium (2,5) becomes a repellor, and an attracting cycle is born that increases in size as b decreases from b_c.

2. Product Bifurcations

The Hopf bifurcation is intrinsically two dimensional. A lot of two-dimensional bifurcations are intrinsically one-dimensional. The simplest of these are the product bifurcations. Given a family of first-order equations, for example,

$$\dot{x} = r + x^2 , \qquad (2)$$

one can form the **product family** by adding the equation $\dot{y} = -y$:

$$\begin{aligned} \frac{dx}{dt} &= r + x^2 \\ \frac{dy}{dt} &= -y \end{aligned} \qquad (3)$$

The family Equation (2) has a 1-D saddle node bifurcation at $r = 0$ (See Lab 23). Knowing the behavior of the family Equation (2) makes it very easy to figure out the behavior of the product family Equation (3). Fix an interesting value of r. Take the x-axis as horizontal and draw the phase line picture for Equation (2). In the upper-half plane, $\frac{dy}{dt} < 0$ so all vectors of the vector field point downward. Above an equilibrium point, the vector field points directly down. Above a point on the phase line with the arrow pointing right, the vector field points down and to the right. Above a point on the phase line with the arrow pointing left, the vector field points down and to the left. In the lower-half plane, the vector field points upward and a similar analysis can be applied.

For fixed r, to draw the phase portrait of Equation (3), first draw the phase line of (2), draw the trajectories directly above and directly below the equilibrium points, and then fill in a representative set of trajectories.

2.1 Draw the phase portrait of the system in Equation (3) with $r = 1$.

2.2 Draw the phase portrait of the system in Equation (3) with $r = -1$.

2.3 Why is the bifurcation in Equation (2) called a saddle-node bifurcation?

Another family exhibiting a saddle-node bifurcation when the parameter r is 0 is

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x^2 - y - r \end{aligned} \tag{4}$$

Open the **2-D Saddle Node Bifurcation** tool to see an animation of how the phase portrait of Equation (4) changes as r varies through 0. Equation (4) does not define a product family, but it is essentially one-dimensional in the sense that its phase portraits for various values of r are the same as the phase portraits for Equation (3) up to "continuous change of coordinates."

2.4 Find all equilibria for Equation (4) when $r = -1$ and when $r = 1$, linearize Equation (4) at the equilibria, and verify how it behaves near the equilibria.

2.5 For the family of first-order differential equations $\dot{x} = r^2 + x^2 - 1$, draw the bifurcation diagram. Form the product family. For a representative set of values of r, fix r and draw the phase portrait of the system. Describe what happens at each bifurcation of the product family.

Note: The Hopf bifurcation and product bifurcations are just two illustrations of the kinds of bifurcations planar autonomous systems can undergo.

Lab 25: Tool Instructions

Chemical Oscillator Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial conditions for a trajectory.
Clicking in the plane while the trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to set the parameter b.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the mouse on the **[Clear]** button to remove all the trajectories from the graph.
Click the mouse on the **[Pause]** button to stop a trajectory without canceling it.
Click the mouse on the **[Continue]** button to resume the motion of the paused trajectory.

2-D Saddle-Node Bifurcation Tool

Buttons

Click on the arrow buttons to control the animation sequence of the path of the parameter k. Use the double arrow buttons to play the sequence forward and backward. Use the single arrow buttons to advance and reverse the sequence one frame at a time.

Click on the single arrow button to stop the sequence.