24 Spruce Budworm

Tools Used in Lab 24

- Spruce Budworm: Time Series
- Spruce Budworm: *kr*-Plane
- Spruce Budworm: *rx*-Plane
- Spruce Budworm: Cusp
- Spruce Budworm: Hysteresis
- Imperfect Bifurcation

The spruce budworm is a serious pest in eastern Canada, where it attacks the leaves of the balsam fir tree. When an outbreak occurs, the budworms can defoliate and kill most of the fir trees in the forest in about four years.

1. Introduction

The equation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2} \qquad (1)$$

gives an idealized population model for the spuce budworm, first proposed and analyzed by Ludwig, Jones, and Holling (*J. Anim. Ecol.* 47 (1978), 315). The variables t and x should be thought of as time and the spruce budworm population, respectively. The first term on the right side models the growth of the spruce budworm population without predation as logistic. The constant parameters $k > 0$ and $r > 0$ should be thought of as the carrying capacity of the forest (maximum number of spruce budworms that can live in the forest) and the growth rate of the spruce budworm population, respectively, in the absence of predation. The second term models the effect of predation, chiefly by birds. For a more detailed discussion of the spruce budworm models, see Steven H. Strogatz, *Nonlinear Dynamics and Chaos*, Addison-Wesley, Reading, MA, 1994.

1.1 What does (i) $\frac{dx}{dt} > 0$, (ii) $\frac{dx}{dt} < 0$, (iii) $\frac{dx}{dt} = 0$ imply about the change of the spruce budworm population with time?

2. Time Series

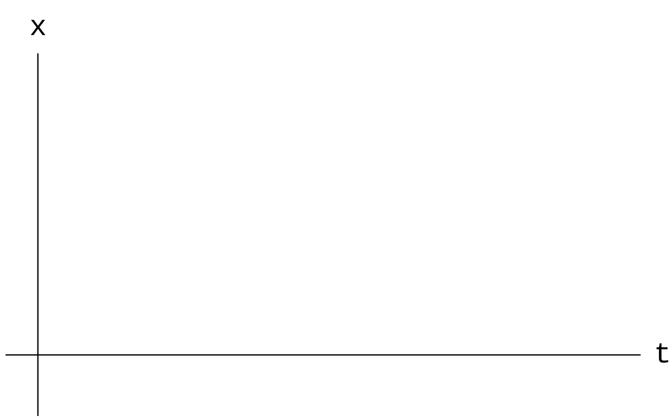
Due to the continuity of the right-hand side of Equation (1), the sign of $\frac{dx}{dt}$ can change only when $\frac{dx}{dt} = 0$ This happens when $x = 0$, or when

$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1 + x^2} \qquad (2)$$

Note that the right-hand side of Equation (2) is independent of the parameters, and the left-hand side of Equation (2) is linear in x. Now open the **Spruce Budworm: Time Series** tool. The two sides of Equation (2), $y = r\left(1 - \frac{x}{k}\right)$ and $y = \frac{x}{1 + x^2}$, are plotted on the same graph. The parameters k and r can be varied using the sliders.

2.1 How can you tell from the graphs of $y = r\left(1 - \frac{x}{k}\right)$ and $y = \frac{x}{1 + x^2}$ whether $\frac{dx}{dt}$ is positive, negative, or zero?

2.2 Find values of k and r for which the right side of Equation (1) has only one zero besides $x = 0$ and sketch the time series for your choice of parameters and representative choices of initial conditions.



2.3 For your choice of r and k in Exercise **2.2** and an initial value of spruce budworm population above the one positive equilibria of the differential equation in Exercise **2.2**, how does the spruce budworm population vary with time? Why?

2.4 Find values of k and r for which the right-side of Equation (1) has three zeros besides $x = 0$ and sketch the time series for your choice of parameters and representative choices of initial conditions.

3. Another Viewpoint

Open the **Spruce Budworm: kr-Plane tool**. You choose values of the parameters k and r by moving the cursor in the kr-plane. The graph of the right-hand side of from Equation (1) is displayed, as well as the graphs of $y = r\left(1 - \frac{x}{k}\right)$ and $y = \frac{x}{1+x^2}$.

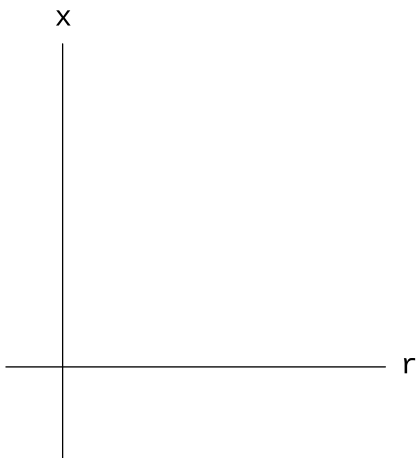
3.1 What is the significance of the cusp in the kr-plane?

4. Bifurcation Diagram

The bifurcation diagram for Equation (1) is the set of points in krx-space for which $\frac{dx}{dt} = 0$. Since $x = 0$ is an equilibrium, each point with $x = 0$ belongs to the bifurcation diagram. The rest of the bifurcation diagram consists of those points satisfying Equation (2).

Open the **Spruce Budworm: rx-Plane** tool. Choose several values of k with the slider and observe the graph of Equation (2) in the rx-plane. For k constant, the bifurcation diagram is this curve in the rx-plane.

4.1 Choose a value of *k* for which Equation (2) does not define *x* as a function of *r* (recall the vertical line test). Draw the bifurcation diagram for this fixed *k* (use a solid curve for those equilibria that are attractors and a dashed curve for those equilibria that are repellors). For a representative set of values of *r*, indicate the long-term behavior of the spruce budworm population *x* by drawing arrows on your bifurcation diagram (moving the cursor in the *rx*-plane chooses a value of *r* and displays the graph of $\frac{dx}{dt}$).



The graph of Equation (2) in *krx*-space can be generated from the graphs of Equation (2) in the *rx*-plane with *k* fixed by allowing *k* to vary. Open the **Spruce Budworm: Cusp** tool to view a graph of Equation (2). The interior of the cusp drawn in the *kr*-plane in the **Spruce Budworm: *kr*-Plane** tool, contains the points in the *kr*-plane covered by 3 points of the graph of Equation (2). That is, for each point (*k*, *r*) within the cusp there are 3 points on the graph of Equation (2) with those *kr*-coordinates.

5. Hysteresis

In modeling the spruce budworm population, we assume a fixed *k* and that the growth rate *r* of the spruce budworm population drifts slowly upward with time (because the forest cover increases) and then decreases as the spruce budworms kill off the trees. Open the **Spruce Budworm: Hysteresis** tool. Choose a value of the parameter *k* by clicking in the *kr*-plane. The software causes *r* to drift automatically. Observe how *x* varies with *r* (and thus with time) for various choices of *k*.

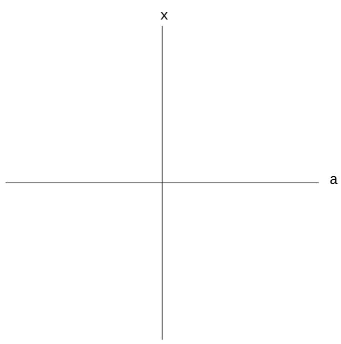
5.1 For the value of *k* you used in Exercise **4.1**, describe and explain this variation of spruce budworm population *x* with *r* and thus with time.

6. A Similar Equation

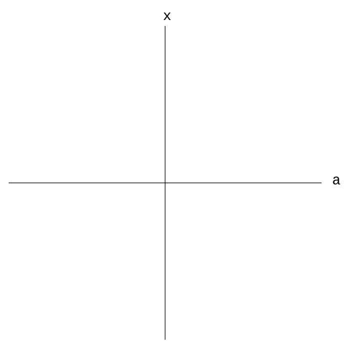
An equation that exhibits dynamics similar to the dynamics of the spruce budworm equation, but with simpler algebra is given by

$$\frac{dx}{dt} = a + rx - x^3 \qquad (3)$$

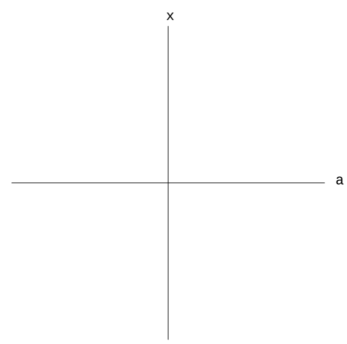
6.1 Fix $r = -1, 0, 1$ in Equation (3) and draw the bifurcation diagrams for the resulting three families of differential equations, each depending only on parameter a. *Hint:* Graph the function $a(x) = x^3 - rx$ and interchange the a-axis and the x-axis so that the x-axis is vertical and the a-axis is horizontal. Remember to dash portions of the curve corresponding to unstable equilibria.



$r = -1$



r = 0

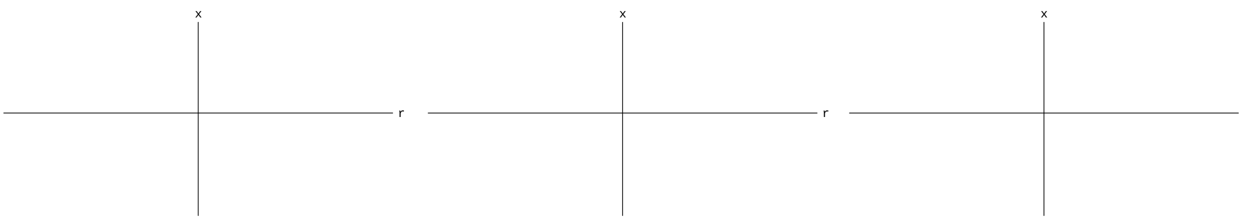


r = 1

You can use these bifurcation diagrams to visualize the entire bifurcation diagram of Equation (3), which consists of the graph of the surface $a + rx - x^3 = 0$ in rax-space.

6.2 If you project this surface onto the ra-plane, there is a cusp that consists of the points in the ra-plane that are images of exactly two points on the surface. Find the equation of this cusp and open the **Imperfect Bifurcation** tool to see this cusp.

6.3 Use the **Imperfect Bifurcation** tool to draw the bifurcation diagrams for the three families of differential equations, depending on the parameter r determined from Equation (3) by fixing $a = -1, 0, 1$. Note that when $a = -1$ or $a = 1$, the bifurcation diagram does not change very much with a small change in a but that when $a = 0$, the bifurcation diagram changes completely with any change in a, no matter how small.



Lab 24: Tool Instructions

Spruce Budworm: Time Series Tool

Setting Initial Conditions

Click the mouse on the graphing plane on the right (labeled *tx*) to see graphical output.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to change the values for the parameters *r* and *k*.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse up and down, or back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Spruce Budworm: *kr*-Plane Tool

Setting Parameters

Move the mouse over the *kr* graph on the left to select values of *r* and *k*.

Spruce Budworm: *rx*-Plane Tool

Setting Initial Conditions

Move the mouse over the *rx* graph on the left to see the graphical output of *dx/dt*.

Parameter Sliders

Use the sliders to change the values for the parameter *k*.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Spruce Budworm: Cusp Tool

This tool is demonstrative, not interactive.

Spruce Budworm: Hysteresis Tool

Setting Initial Conditions

Click the mouse on the *kr* graph on the left to select a value of *k*.

Imperfect Bifurcation Tool

Setting Initial Conditions

Click the mouse on the $x\dot{x}$ graph on the top left, the graphing plane of the lower left (labeled *x*), and the *ra* graphing plane on the top right to see the graphical output.
Clicking in the *tx* plane while the trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to change the values for the parameters *a* and *r*.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse up and down, or back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the mouse on the **[Draw Field]** button to draw a slope field.
Click the mouse on the **[Clear]** button to remove all trajectories from the graph.