23 Bifurcations in 1-D

Tools Used in Lab 23

- Saddle-Node Bifurcation
- Transcritical Bifurcation
- Pitchfork Bifurcation: Supercritical
- Pitchfork Bifurcation: Subcritical

Water suddenly freezes into ice as the temperature is lowered below the freezing point. This is an example of a bifurcation: a qualitative change in the behavior of a system as a parameter is varied.

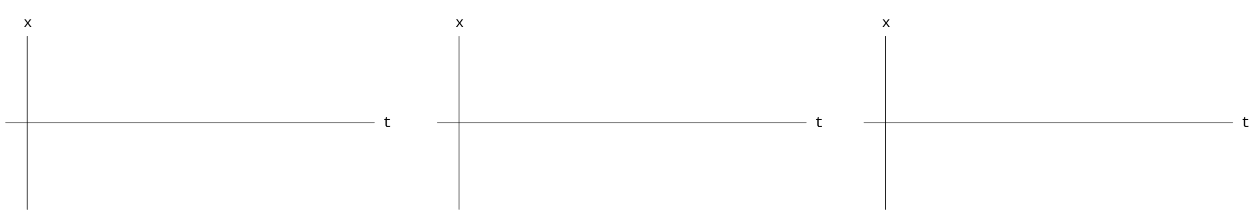
1. Bifurcation

For an autonomous differential equation $x' = f(x,r)$ with one parameter r, changes in the behavior of the solutions due to a small change in r are usually gradual and continuous. At certain values of r, however, the tx pictures may exhibit a sudden *qualitative* change, which is called a **bifurcation.** This lab allows you to explore just how the gradual changes and drastic changes can occur simultaneously.

There are four common ways in which bifurcation occurs in a family of one-dimensional equations $x' = f(x,r)$, where r is a parameter. The four tools for this lab give one example for each case: **Saddle-Node Bifurcation, Transcritical Bifurcation, Pitchfork Bifurcation: Supercritical, Pitchfork Bifurcation: Subcritical.**

We will examine each of these in turn later in the lab, but for now we just want to show what bifurcation means in the tx-plane. The four tools all work the same way.

1.1 To see an example of bifurcation, open any one of the tools for this lab. By clicking with the mouse, draw some representative solutions in the tx-plane and sketch the results on the appropriate graph. Then repeat, using both a positive value for r and a negative value.



$r < 0$ $r = 0$ $r > 0$

Name of tool used: Equation:

1.2 Describe in words how the behavior of the solutions $x(t)$ has changed as r moves from left to right in Exercise **1.1**:

1.3 Repeat the experiment of Exercise **1.1** for a second tool:

$r < 0$ $r = 0$ $r > 0$

Name of tool used: Equation:

1.4 Describe in words how the behavior of the solutions $x(t)$ has changed as r moves from left to right in Exercise **1.3**:

1.5 Describe the differences in behavior between your two examples from Exercises **1.1** and **1.3**:

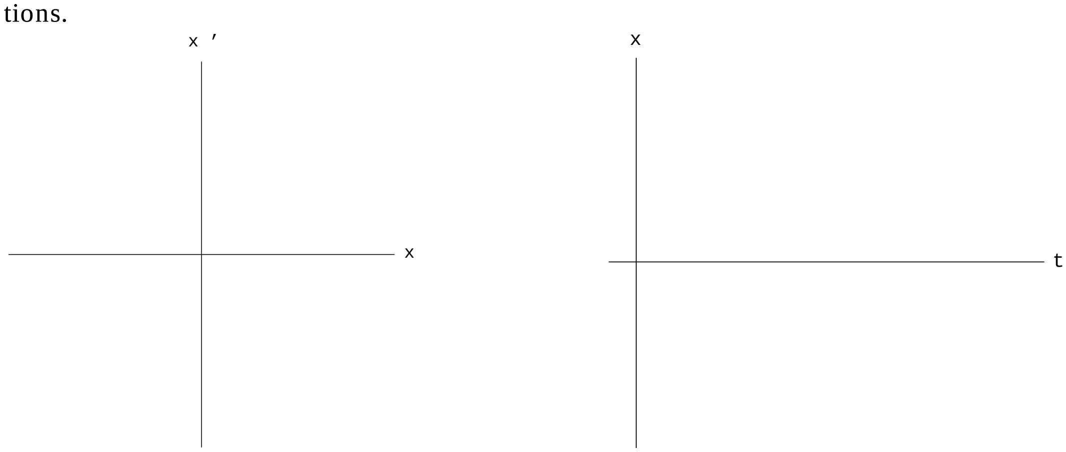
These first exercises should have alerted you to the phenomena we wish to study. Just what is it that changing r does? Before we move on to the details in each of the four types of bifurcation, we need to look at the three graphs in each tool and see how they relate to each other. The **phase line** is the key that ties them all together, and is the focus of the answer to the question we just asked. Although the phase line was discussed in Lab 3, Sections 3 and 4, most textbooks have not given it or the graphs in the xx'- and rx- planes the necessary prominence to serve as a foundation for analyzing bifurcations.

2. The Phase Line and its Role in *xx'* and *rx* Graphs

Once a value of *r* has been chosen, the *xx'* graph on the upper left can be drawn from the differential equation; the phase line forms its horizontal axis. The points at which the graph crosses the *x*-axis are equilibrium points, where $x' = 0$. The points where *x'* is positive are where solutions are moving in the direction of increasing *x*; where *x'* is negative, solutions are moving in the direction of decreasing *x*—this information determines the directions of the arrowheads on the phase line.

The same phase line is rotated to a vertical position in the *tx* graph in the lower left of the screen.

2.1 Using any of the four tools, sketch an *xx'* graph and the corresponding *tx* graph, with some solutions.



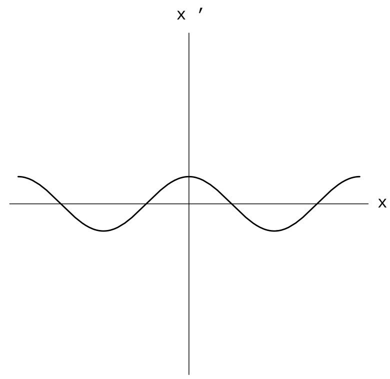
2.2 Mark the phase line in the graphs for Exercise **2.1**. It is the horizontal axis on the *xx'* graph and the vertical axis on the *tx* graph. These should be marked with exactly the same configuration in the *x* direction in each case, with marked equilibrium points and appropriate arrows between. The convention is to color an equilibrium point solid if it is *stable* (that is, if arrowheads on either side are coming toward it), to leave it open if the equilibrium is *unstable* (both arrowheads point away).

2.3 How might an equilibrium be neither stable nor unstable? Support your answer with a picture of a possible case.

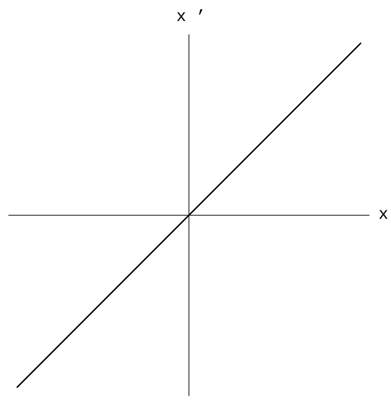
2.4. What behaviors on the *tx* graph correspond to the points of equilibrium on the phase line? to the arrows pointing up on the phase line? to the arrows pointing down?

2.5 To emphasize that the qualitative features of the *tx* graph can be inferred from the *xx*' graph, draw a *tx* graph for each of the following *xx*' graphs. Start by drawing the horizontal phase line on the *xx*' graph, then the same thing vertically on the *tx* graph; then sketch some sample solutions from the information contained in the phase line. Pay attention to whether a given solution $x(t)$ is increasing, decreasing, or constant. Try also to get the proper concavity: if a solution is rising, is it getting steeper as it rises (accelerating) or is it getting shallower (decelerating)?

a.

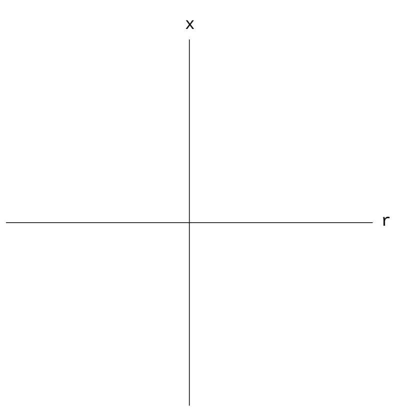


b.

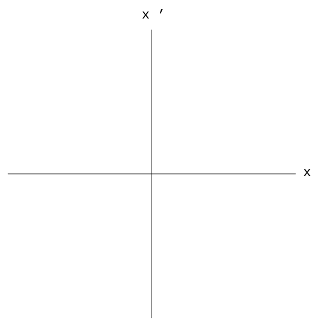


The third graph in each tool is composed of adjacent vertical phase lines. That is, on the *rx* graph, for each value of *r* you could plot a phase line vertically. The locus followed by a stable equilibrium as it moves when *r* changes is drawn as a solid curve; the locus followed by an unstable equilibrium as it moves when *r* changes is drawn as a dotted curve. The result is called a **bifurcation diagram** because it summarizes for any equation $x' = f(x,r)$ what you can expect for a phase line (and hence for the behavior of solutions in the *tx* plane) for any *r*.

2.6 For any of the tools, sketch here the *rx* bifurcation diagram.

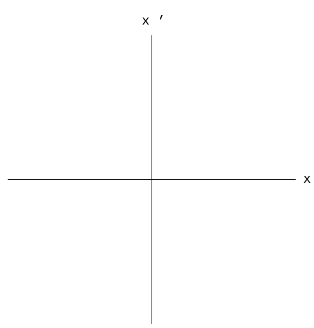


Now you will demonstrate how to obtain key aspects of the other graphs from this one, without the computer! Without using the tool further, choose a value of *r* other than that where your example is set, for instance *r* = –1, and sketch by hand the critical features of the *xx'* graph—not from the equation, but just from the phase line information you can read from the *rx* graph at that *r* value. That is, lay along the horizontal axis of the *xx'* graph the phase line information with appropriate arrows, since you know which equilibrium is stable and which is unstable by whether it lies on a solid or dotted bifurcation curve. Where the arrows point to the right, sketch *x'* above the horizontal axis, and where they point to the left, sketch *x'* below the axis.



This should give a very good qualitative sketch, and from that you can sketch a *tx* graph.

2.7 Return to the computer tool, input your new *r* value, and sketch below the *xx'* and *tx* graphs that result. You can expect that the details of how high the computer *xx'* graph rises above or below the horizontal axis will not usually match your hand sketch of the same graph, but the qualitative features of where the graph crosses the axis, and how, should be exactly the same.



Explain the differences that may occur between your computer and hand sketches of the *tx* graph:

At this point you should have a pretty good understanding of all the graphs in each tool and how they relate to one another. It is time to look individually at each of the four kinds of bifurcation for a one-dimensional differential equation, keeping in mind that the overriding question is "What does changing *r* do to the fixed points and the phase line?"

3. Saddle-Node Bifurcation

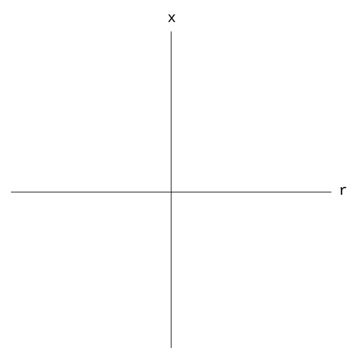
The special feature of saddle-node bifurcation is that as *r* changes in one direction or the other, the number of equilibria changes—you should note the number and the type in each example.

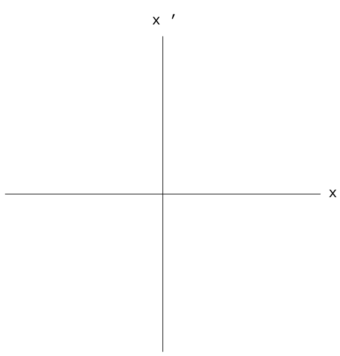
The name “saddle-node” comes from the two-dimensional analog, but you can think of “saddle” as an unstable fixed point, and “node” as a stable one.

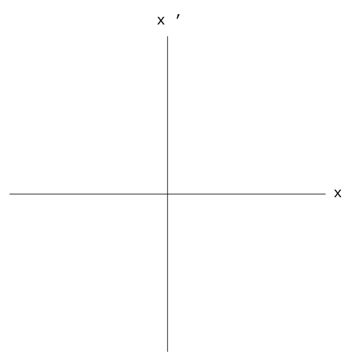
3.1 A nice standard example of a saddle-node bifurcation is given by

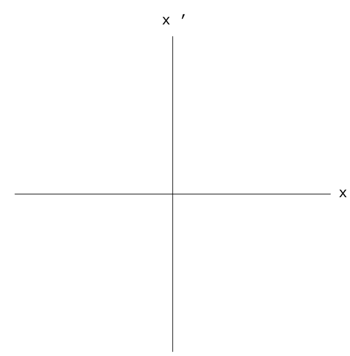
$$\dot{x} = r + x^2. \qquad (1)$$

Open the **Saddle-Node Bifurcation** tool and experiment with different values of *r*, if you haven’t already, to confirm this definition. Sketch the *rx* bifurcation diagram, and sketch a typical set of behaviors for a negative, zero, and positive value of *r*. Explain what happens to the equilibria on the phase line as *r* moves from left to right.









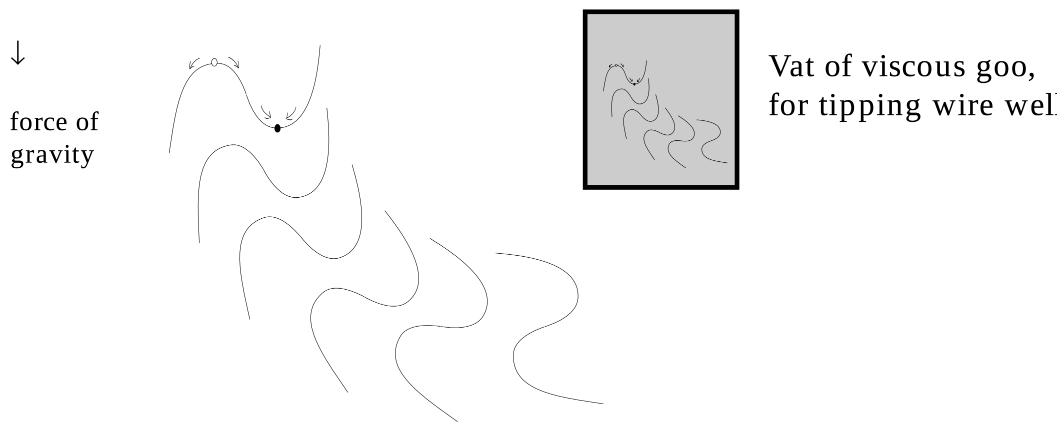
$r = -$____ $r = 0$ $r = +$____

Explanation:

3.2 A nice example of a saddle-node bifurcation comes from tipping a wire that forms a well. Imagine a bead on this wire (and to be physically correct, you should imagine the whole apparatus in a vat of viscous goo—this provides viscous damping, so that whenever possible, the bead would tend to settle in the bottom of the well).

In the topmost curve, the black circle represents a stable equilibrium for the bead; whenever it is perturbed a bit, it will settle back down to the bottom of the well. The open circle represents another equilibrium; if the bead were perfectly balanced on this peak, it would stay there, but if it's even slightly perturbed, it will roll downhill away from the peak—hence this equilibrium is unstable. The arrows show the directions a bead would slide on each segment of the wire.

For each of the subsequent positions of the wire, draw the location of the stable and unstable equilibria—the bottom of the well and the top of the peak from the point of view of gravity directed straight down the page. Color the stable equilibrium black and leave the unstable one open. Draw the arrows as well. What happens?



3.3 For $x' = r - x - e^{-x}$, find where the bifurcation is located, and sketch representative *xx'*, *rx*, and *tx* graphs. This should be done on a separate sheet of paper. *Hint*: Since the *xx'* graph is awkward to draw by hand, an easy trick is to graph separately $r - x$ and e^{-x} on the same axes; the intersections show the points (r,x) where the equilibria are located. You can plot this for several different r values, and then make *rx* and *tx* graphs from the information you can glean. This exercise can be done qualitatively by hand, or more precisely using open-ended graphing tools.

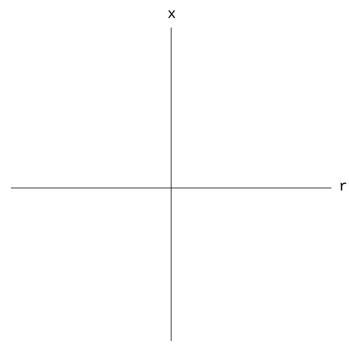
4. Transcritical Bifurcation

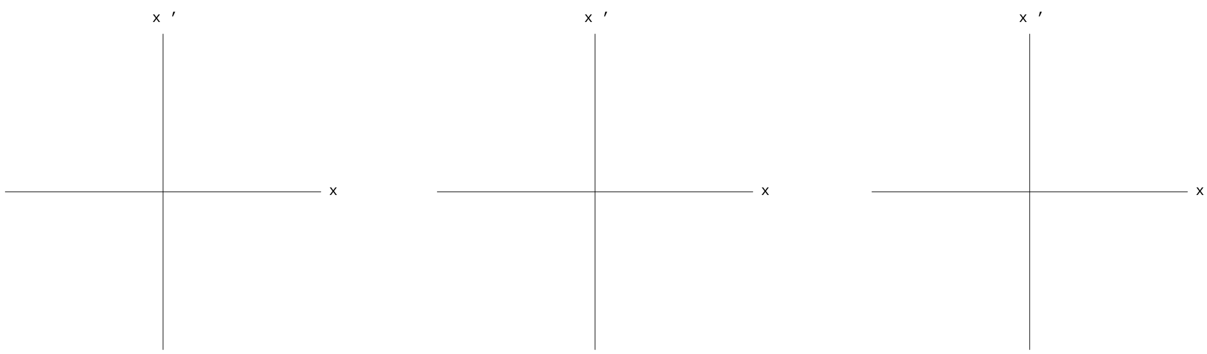
In a transcritical bifurcation, the key is that two equilibria change stability. They don't disappear.

4.1 The standard transcritical bifurcation generally encountered is provided by

$$\dot{x} = rx - x^2. \qquad (2)$$

Open the **Transcritical Bifurcation** tool, sketch the *rx* bifurcation diagram, and experiment with different values of *r*. Explain, with sketches of the *xx'* plane and the corresponding *tx* sketches, how it is that the two equilibria exchanged stabilities.





$r = -$____ $r = 0$ $r = +$____

Explanation:

4.2 Locate the bifurcation value for *r* in the differential equation $x' = rx - x(1 - x)$ on a separate sheet of paper. By hand or with open-ended computer tools, sketch the *xx'*, *rx*, and typical *tx* graphs.

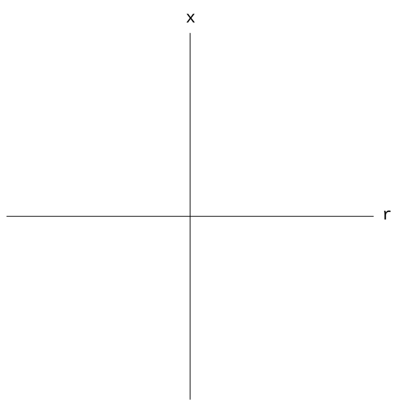
5. Pitchfork Bifurcation

A pitchfork bifurcation arises when certain symmetries exist; its bifurcation diagram in the *rx* plane looks like a pitchfork, with three tines and a single handle. Although a convention has arisen to distinguish two subtypes of pitchfork bifurcation, supercritical and subcritical, some general observations can be made about the pitchforks in both cases.

5.1 From the tools **Pitchfork Bifurcations**: **Supercritical** and **Pitchfork Bifurcations**: **Subcritical**, sketch the *rx* bifurcation diagrams.

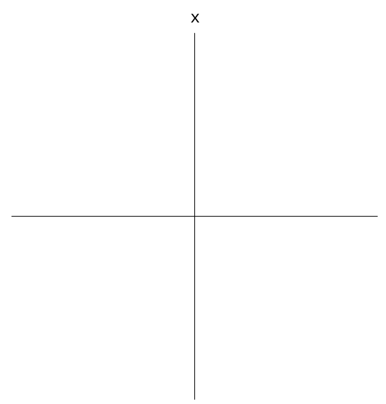
Supercritical Pitchfork

$x' = rx - x^3$



Subcritical Pitchfork

$x' = rx + x^3$



5.2 What happens to the stability of the handle when it becomes the middle tine of the pitchfork?

5.3 The outer two tines of the pitchfork represent a symmetric pair of equilibria on one side or the other of the bifurcation value. What kinds of stability do they represent?

5.4 Describe the differences between supercritical and subcritical pitchfork bifurcations.

5.5 For each of the two types of pitchfork bifurcations, from the *rx* diagrams in **Exercise 5.1**, predict, by hand, the *tx* behaviors for the various cases.

Supercritical Pitchfork

$r < 0$

$r = 0$

$r > 0$

Subcritical Pitchfork

$r < 0$

$r = 0$

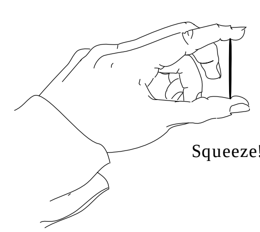
$r > 0$

5.6 With the **Pitchfork Bifurcation: Supercritical** and **Pitchfork Bifurcation: Subcritical** tools, confirm your sketches in Exercise **5.5**. Explain any differences, and distinguish between the two subtypes.

5.7 On a separate sheet of paper, locate the bifurcation value(s) for r in the equation $x' = rx + x/(1+x^2)$. Sketch *xx'*, *rx*, and typical *tx* graphs.

6. Simple Physical Examples

One very simple example of bifurcation is the shape of a thin plastic coffee stirrer being squeezed when held by its ends between thumb and forefinger. Eventually the stirrer will buckle, to one side or the other. In a symmetrical situation, buckling to either side is equally likely, so this is a nice example of a pitchfork bifurcation.



6.1 Would this be a subcritical or supercritical pitchfork bifurcation? Why?

Population models are another source of bifurcation behavior. A good example is provided by the harvesting example of Lab 3 on single species population models.

6.2 Open the **Logistics with Harvest** tool of Lab 3, Section 4. Here $x' = x(1 - x) - h$. At what value of h does bifurcation occur?

6.3 What is the nature of the bifurcation of Exercise **6.2** when the harvesting becomes too intense? Name it mathematically, but describe it biologically.

6.4 We have not readily thought up a simple example of a transcritical bifurcation. Can you? We would be happy to use the best submitted to us in our next edition, and give you credit.

7. Additional Exercises

You can now try to find and identify bifurcations from equations in general with a parameter.

7.1 Make a chart that compares and classifies the three basic types of bifurcation for one-dimensional equations $x' = f(x,r)$, based on your observations in Sections 3, 4, and 5. What properties are special to each type? How might you sum up the identifying criteria for each type?

	Saddle-Node	Transcritical	Pitchfork
How do xx' graphs change as r changes?			
How do tx graphs change as r changes?			
Draw a typical rx graph, summarizing all the phase lines.			
Identifying criteria:			

7.2 For the following equations, identify which kinds of bifurcations can occur, and where. ("None" is a possible correct answer.) You can do it by hand or by using a computer with open-ended graphing tools, and you can work from whatever information you find easiest to get in each case. Clearly describe (on a separate page) your steps and your reasoning. In some cases you may find more than one bifurcation, and they might be of different types; in some cases you may find additional phenomena to watch out for or comment upon.

a. $x' = r + \cos(x)$

b. $x' = r + x/2 - x/(1 + x)$

c. $x' = rx \sin x$, for $r > 0$

d. $x' = rx - \sin x$

Lab 23: Tool Instructions

Saddle-Node Bifurcation Tool

Setting Initial Conditions

Click the mouse on the $x\dot{x}$ graph on the top left or the tx graphing plane on the lower left to see graphical output.
Clicking while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to change the values for the parameter r.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the mouse on the **[Draw Field]** button to draw a slope field.
Click the mouse on the **[Clear]** button to remove all trajectories from the graph.

Transcritical Bifurcation Tool

Setting Initial Conditions

Click the mouse on the $x\dot{x}$ graph on the top left or the tx graph on the lower left to see graphical output.
Clicking while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to change the values for the parameter r.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the mouse on the **[Draw Field]** button to draw a slope field.
Click the mouse on the **[Clear]** button to remove all trajectories from the graph.

Pitchfork Bifurcation: Supercritical Tool

Setting Initial Conditions

Click the mouse on the $x\dot{x}$ graph on the top left or the tx graphing plane on the lower left to see graphical output.
Clicking while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to change the values for the parameter r.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the mouse on the **[Draw Field]** button to draw a slope field.
Click the mouse on the **[Clear]** button to remove all trajectories from the graphs.

Pitchfork Bifurcation: Subcritical Tool

Setting Initial Conditions

Click the mouse on the $x\dot{x}$ graph on the top left or the *tx* graphing plane on the lower left to see graphical output.

Clicking while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to change the values for the parameter *r*.

Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

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