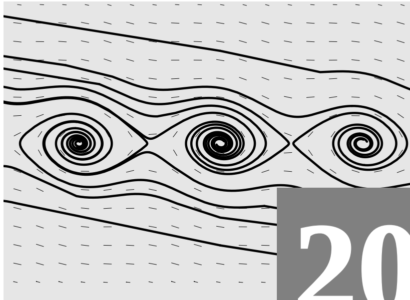


20 Nonlinear Oscillators: Free Response

Tools Used in Lab 20
Pendulums

The child on the backyard swing wants you to start her as high as you can manage with as much initial angular velocity as you can muster. Assumptions about small angular displacements no longer hold. Is there a loop-the-loop in her future?

1. The Undamped Pendulum

The equation for the free (unforced) pendulum without damping is

$$\frac{d^2\theta}{dt} + \frac{g}{L}\sin\theta = 0\,, \qquad (1)$$

where g is the acceleration of gravity, θ, a function of time t, is the angular displacement from the downward vertical, and L is the length of the pendulum. We define $\omega \equiv \sqrt{g\,/\,L}$ to simplify Equation (1), which becomes

$$\frac{d^2\theta}{dt} + \omega^2\sin\theta = 0\,. \qquad (2)$$

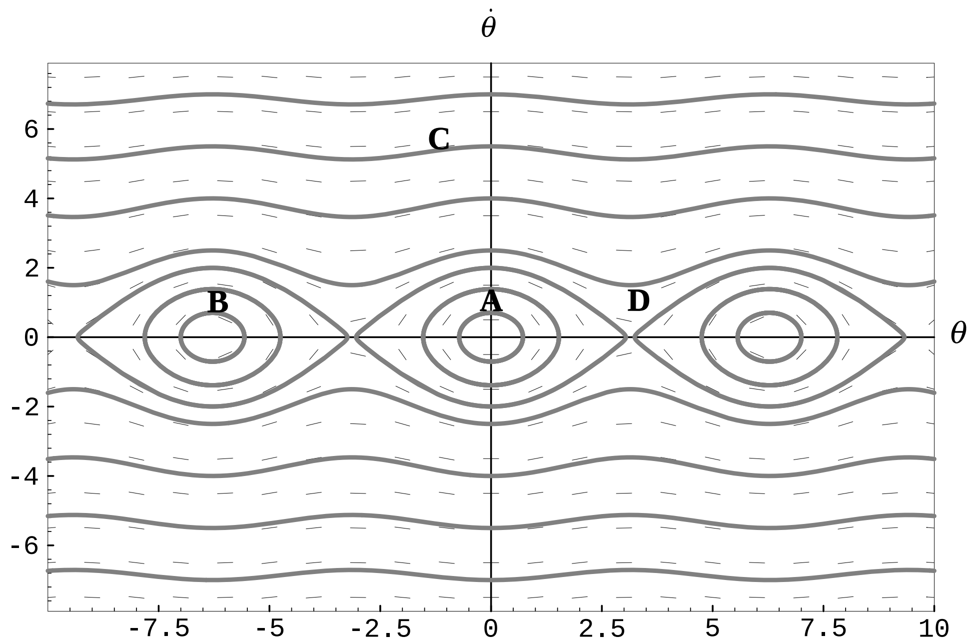
1.1 The Linear Simplification

For small values of θ, the values for θ and $\sin\theta$ radians are nearly equal. Use this fact to simplify Equation (1). Then solve the new linear equation to obtain the formula for the resulting simple harmonic motion. Show that the frequency in radians per second is the frequency ω defined above.

This motion is illustrated in the **Pendulums** tool in the **Linear** option. Now that you have reviewed the linear model, move on to the real thing, the nonlinear pendulum. Select the undamped, unforced **Simple Nonlinear** option. Select some nontrivial initial conditions and observe the resulting phase plane trajectories and time series plots for the angle, θ, and the angular velocity, $\dot{\theta}$.

1.2 The Phase Plane Portrait for the Free Undamped Pendulum

In your responses to the following questions on the pendulum motions that produced the phase plane below, include observations about closed orbits, simple harmonic motion, unstable and / or stable equilibria, and, of course, loop-the-loops or rotations.



a. Discuss the motion of the pendulum that results in trajectory A.

b. Orbits A and B have the same size and shape; both orbits are the smallest shown and are horizontal translates of each other. Discuss the motion of the pendulum that results in trajectory B. How does it differ from trajectory A, if at all?

c. Discuss the motion of the pendulum that results in trajectory C.

d. Describe what happens to the pendulum at the saddle near D.

2. The Damped Unforced Pendulum

If we include viscous damping with damping constant *b*, the equation becomes

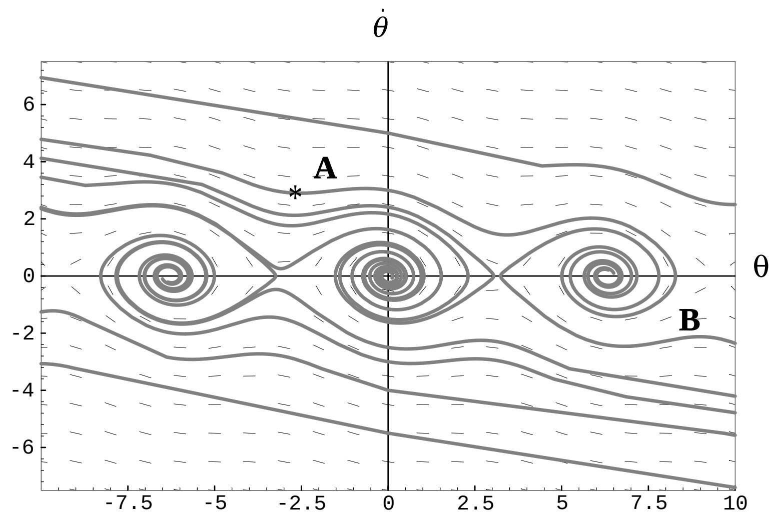
$$\ddot{\theta} + b\dot{\theta} + \omega^2 \sin\theta = 0 \qquad (3)$$

2.1 Rewrite Equation (3) as a linear system of first-order differential equations for angular velocity, $\dot{\theta} = \mu$ and angular acceleration, $\dot{\mu}$.

$\dot{\theta} = \mu$

$\dot{\mu} =$

2.2 A phase plane diagram for the damped pendulum is shown for $b = 0.2$ and $\omega = 1$.



a. Carefully describe the motion of the pendulum that corresponds to trajectory A in the preceding graph. Assume that A signifies the beginning of the trajectory at time $t = 0$. Be sure to start with your estimation of the signs of initial conditions $\theta(0)$ and $\dot{\theta}(0)$.

b. Make the same sort of analysis of trajectory B. What is happening to the pendulum?

3. Further Exploration

3.1 Take a look at the **Damped Nonlinear** and **Forced Damped Pendulum** options, and describe what is different about the behaviors in each case. This is studied further in Lab 26, Chaos in Forced Nonlinear Oscillators.

Lab 20: Tool Instructions

Pendulums Tool

Setting Initial Conditions

Click the mouse on the $\theta\dot{\theta}$ plane to set the initial conditions for a trajectory.
Clicking while a trajectory is being drawn will start a new trajectory.

Time Series Buttons

The buttons labeled
[ ] **position**
[ ] **velocity**
[ ] **acceleration**
toggle the time series graphs on and off.

Other Buttons

Click the mouse on **[Linear (SHO)]**, **[Simple Nonlinear]**, **[Damped Nonlinear]**, or **[Forced Damped]** buttons to select a pendulum model.
Click the **[Pause]** button to stop a trajectory without canceling it.
Click the **[Continue]** button to resume the motion of a paused trajectory.
Click the mouse on the **[Clear]** button to remove all trajectories and vectors from the graph.