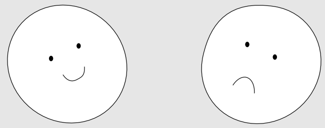

18 Romeo and Juliet

Tools Used in Lab 18
Romeo and Juliet

Every love affair has its ups and downs over time . . . so can it be modeled by differential equations?

Disclaimer: Only the names are the same.

1. The General Case

The situation is fictional but perhaps familiar. The gender of the participants can be changed to fit personal situations. The problem was invented by Steven Strogatz [SS1], the “treatment program” instituted by Bjørn Felsager [MF], and further alterations and inventions can be made by you after you understand the nature of the problem. You will see that love is sometimes best modeled by an oscillator . . . perhaps, alas, a damped oscillator.

Although love by its nature may be nonlinear, we restrict our attention to the linear case. This leaves the door open for further research.

Let x and y be functions of time, where x denotes Romeo’s love for Juliet and y denotes Juliet’s love for Romeo.

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

where a, b, c, and d are constants.

Note that these equations can be written in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Setting the Stage

Romeo's love for Juliet cools in proportion to her love for him. Juliet's love for Romeo grows in proportion to his love for her. Time is measured in days (0–50) and their love is measured on a scale from –5 to 5, where 0 is indifference.

Hysterical hatred	Disgust	Indifference	Sweet attraction	Ecstatic love
–5	–2.5	0	2.5	5

When first they meet, Romeo is immediately attracted to Juliet, but she is as yet indifferent. Soon, however, the tide will turn . . .

The **Romeo and Juliet** tool allows you to change elements in the matrix by clicking on the arrows beside the values to make them larger or smaller. You can set the initial conditions by clicking on an appropriate spot in the phase plane. The initial conditions described above are $x(0) = 2$, $y(0) = 0$. Note that the form of the equations given in the tool uses $(x + h)$ and $(y + k)$ instead of x and y. This is useful for Exercise **3.1**.

1.1 Set the matrix entries in the **Romeo and Juliet** tool to correspond to the system of equations below. Use initial conditions $x(0) = 2$, $y(0) = 0$.

$$\frac{dx}{dt} = -0.2y$$

$$\frac{dy}{dt} = 0.8x$$

a. Look at the phase plane and time series for x and y to analyze the ensuing relationship. How would you characterize their relationship?

b. Look at the phase plane. What percentage of the time do they experience mutually positive feelings for each other?

c. Examine the time series for x and y. What are Juliet's feelings for Romeo when he is most attracted to her?

What are Romeo's feelings for Juliet when she is most attracted to him?

d. What is the maximum attraction or love that Romeo feels for Juliet? What is the maximum attraction or love that she feels for him? Give your answers in both numbers and words.

e. What is the period of their emotional cycles? Do they have the same period?

f. What is the time between Romeo's and Juliet's emotional peaks (that is, the shortest time)?

2. The story continues . . .

Juliet's ride on an emotional roller coaster begins to disorient her. She is distracted, losing sleep and forgetting to do her homework, so she goes to a counselor for advice. The counselor decides that she is over-responding to the emotional stimuli and puts her on a tranquilizer. The new equations follow.

$$\frac{dx}{dt} = -0.2y$$

$$\frac{dy}{dt} = 0.8x - 0.1y$$

2.1 Adjust the matrix entries to reflect the new situation. Use the same initial conditions as in Exercise **1.1**: ($x(0) = 2$, $y(0) = 0$).

a. Sketch the phase plane and the time series. What happens? Can we say that the tranquilizer has a damping effect on the relationship?

b. Suppose the counselor now tries instead to increase the responsiveness of Romeo to Juliet by giving him a stimulant. What do you expect to happen to the relationship? Using the equations below and the same initial conditions, does the graph confirm your prediction?

$$\frac{dx}{dt} = -0.2y + 0.1x$$

$$\frac{dy}{dt} = 0.8x$$

3. And the story continues . . .

Romeo and Juliet are appalled at the changes in their relationship and immediately change counselors. The new counselor, wiser and more understanding, realizes that a shift in attitudes must take place. They stop taking pills and start to learn new patterns of response to each other. Romeo learns to accept Juliet's love and now his love begins to decrease only if she becomes overly affectionate ($y > 2$). Juliet learns to control her responses and her love grows only when Romeo becomes very affectionate ($x > 2$). The equations become

$$\frac{dx}{dt} = -0.2(y - 2)$$

$$\frac{dy}{dt} = 0.8(x - 2)$$

3.1 Change the matrix entries accordingly and the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x-2 \\ y-2 \end{bmatrix}$. Use the same initial conditions. Sketch the phase portrait and time series. Describe what happens. Have they found happiness at last?

4. Variations on the Theme

Of course, there are many variations on the modeling of love affairs with differential equations. We can include time dependence, such as the onset of spring, or triangulate with a new love or an old flame.

The **Romeo and Juliet** tool is designed to handle only linear equations, but within these limitations there are many variations. Some of these variations are included in the exercises that follow. For further variations, consult the references listed at the end of the lab.

4.1 One of the references [SS2, 138-144] examines the effects of the signs of the parameters on the course of the love affairs. Pick one of the following "cases," sketch a typical phase portrait, and write a short analysis of the evolution of the resulting relationship.

a. If each lover is disconcerted by the other's emotions ($b < 0$, $c < 0$) but is excited by his or her own feelings ($a > 0$, $d > 0$), what would be the course of their love affair? Pick some values for the parameters and find out.

$\dot{x} =$

$\dot{y} =$

b. Suppose Juliet's love is constant ($\dot{y} = 0$) no matter what Romeo feels. What would the system of equations look like? Examine some cases.

$\dot{x} =$

$\dot{y} =$

c. Do likes attract? Suppose the two lovers had exactly the same emotional profile in terms of their response to each other and their response to their own feelings. Investigate some situations and write a short analysis.

$\dot{x} = ax + by$

$\dot{y} = bx + ay$

d. Suppose Romeo and Juliet are both enthusiastic ($a, b, c, d > 0$) or both cautious ($a, b, c, d < 0$). What is the course of their love affair? Suppose one is cautious and one is enthusiastic? It is interesting to compare using the same initial conditions.

both enthusiastic:

$\dot{x} =$

$\dot{y} =$

both cautious:

$\dot{x} =$

$\dot{y} =$

one cautious, one enthusiastic:

$\dot{x} =$

$\dot{y} =$

References

[**MF**] McDill, J.M. and Bjørn Felsager. "The Lighter Side of Differential Equations," *College Mathematics Journal* 25 (November 1994): 448–452.

[**SS1**] Strogatz, Steven. "Love Affairs and Differential Equations" *Mathematics Magazine* 61 (February 1988): 35.

[**SS2**] Strogatz, Steven. *Nonlinear Dynamics and Chaos, with Applications to Physics, Biology, Chemistry and Engineering.* Reading: Addison-Wesley, 1994.

Lab 18: Tool Instructions

Romeo and Juliet Tool

Setting Initial Conditions

Click the mouse on the *xy* graphing plane to set the initial conditions for a trajectory.
Clicking while a trajectory is being drawn will start a new trajectory.

Matrix Element Values

Click the arrow buttons to the left and right of the matrix elements to increase and decrease their values respectively.

Buttons

Click the mouse on the **[Draw Field]** button to draw a grid of vectors over the *xy*-plane.
Click the **[Pause]** button to stop a trajectory without canceling it.
Click the **[Continue]** button to resume the motion of a paused trajectory.
Click the mouse on the **[Clear]** button to remove all trajectories and vectors from the graphs.