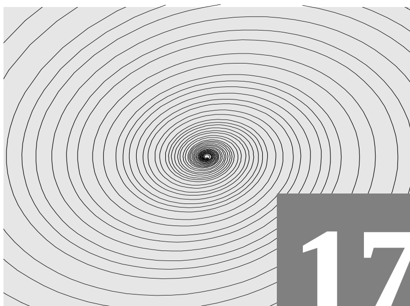


17 Graphing Two-Dimensional Equations

Tools Used in Lab 17

- Simple Harmonic Oscillator
- Parametric to Cartesian
- Phase Plane Drawing
- Vector Fields
- Two Dimensional Equations
- Lotka-Volterra

What do second-order equations have in common with systems of two first-order differential equations? Why are phase planes and vector fields so important? How do they relate to x(t) and y(t) time series?

Second-order differential equations and systems of two first-order equations are both problems in two dimensions. If a solution can be written as an equation, you need a two-parameter family of equations to describe all the solutions. Each individual solution must be specified by a two-dimensional condition, usually an initial condition.

For a second-order equation such as $mx'' + bx' + kx = f(t$, an initial condition would describe both $x(0)$ and $x'(0)$.

For a system of two first-order equations such as $dx/dt = f(x,y)$, $dy/dt = g(x,y)$, an initial condition would describe both $x(0)$ and $y(0)$.

Any second-order equation can be written as a system by assigning $x' = y$ and then noting that $y' = x''$. Hence for our sample equation above, $y' = f/m - by/m - kx/m$. This procedure gives x' and y' each as functions of x and y.

Many systems can similarly be transformed to a second-order equation by solving for one variable in terms of the others and substituting; for example, if you can solve the first equation for y in terms of x and x', then you can substitute in the second for y and y', giving a resulting equation in x'', x', and x.

You can deal with these manipulations outside of the lab. Our purpose here is just to stress that these two problems are essentially the same, so that we can explore in a single discussion the various graphs that result, and how they relate.

1. The Graphs

1.1 The most famous second-order equation is also one of the simplest: $x'' = -x$, describing simple harmonic motion. You can review it with the **Simple Harmonic Oscillator** tool in Lab 9, Linear Oscillators: Free Response.

a. What are the analytic solutions to $x'' = -x$?

b. How do you write this equation as a system of first-order equations?

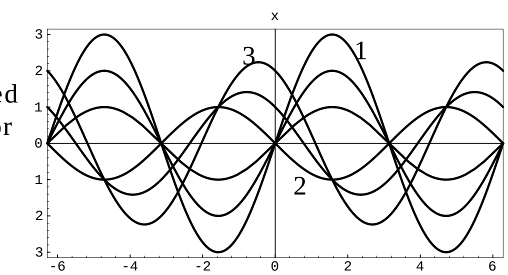
Note: From this point on, we will use the *y* label, with the understanding that in the case of a second-order equation, you can replace *x'* by *y*.

1.2 As you have seen in the labs of Part 2 for second-order equations, we can no longer just draw solutions $x(t)$ from a slope field, as we did for first-order equations. One way to write the analytic solution is

$$x(t) = A\cos(t + B). \quad (1)$$

a. Confirm algebraically that this is equivalent to your answer to Exercise **1.1a**.

b. If you want to graph more than one $x(t)$ solution at a time, the *tx* graph quickly gets complicated. Confirm that this *tx* graph of just a few of these solutions indeed matches Equation (1), by estimating *A* and *B* values for the numbered solution curves.

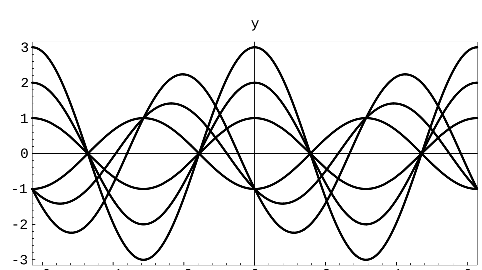


(1) $A_1 \approx$ $B_1 \approx$
(2) $A_2 \approx$ $B_2 \approx$
(3) $A_3 \approx$ $B_3 \approx$

What is the common period of all the curves on the *tx* graph?

c. Differentiate Equation (1) to get a formula for $y(t) = x'(t)$ in this example.

d. Confirm that this *ty* graph comes from the *tx* graph by matching a couple of curves. That is, use your estimated (A,B) values to state the corresponding initial conditions, and identify each of the three associated *x'* curves on the second graph.



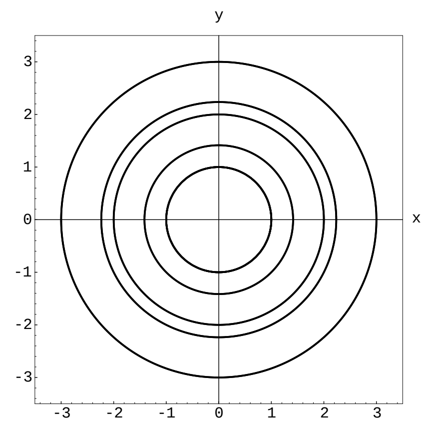
(1) $x(0) =$ $y(0) =$
(2) $x(0) =$ $y(0) =$
(3) $x(0) =$ $y(0) =$

Alternatively, you can accomplish the matching by using the fact that the $y(t)$ curves give the slopes of the $x(t)$ curves, at each value of t.

What period(s) are represented by the $y(t)$ curves?

1.3 Observe that on the *tx* and *ty* graphs, just a few solutions make quite a mess. But if you put the *tx* and *ty* information together as parametric equations, you can get a useful *xy* graph, called the **phase portrait**. Open the **Parametric to Cartesian** tool for a careful demonstration of how this is done for a single solution.

This graph, the *xy* graph associated with the *tx* and *ty* graphs in Exercise **1.2**, is much cleaner!



a. Add arrowheads to show the proper direction of the trajectories, which can be determined directly from the differential equations.

b. You may note, however, that one less curve appears on this phase portrait than in the *tx* and *ty* graphs illustrated. Which two *tx* curves give the same *xy* curve? Mark them with a colored pencil in Exercise **1.2** and do the same for the corresponding *ty* curves.

c. Consider the advantages and disadvantages of each of the three graphs presented. Each is most useful for different purposes. List at least two advantages and one disadvantage for each case.

Pros and cons of *xy*:

Pros and cons of *tx* and *ty*:

2. Making the Phase Plane

Note that the independent variable t does not show explicitly on the phase plane. Experiment with the **Phase Plane Drawing** tool to help answer the question "Where is t?" This tool allows you to draw a curve on the *xy*-plane using the mouse; as you draw, the corresponding $x(t)$ and $y(t)$ graphs appear.

2.1 a. Why is the *tx*-plane at the lower left rotated from its normal position?

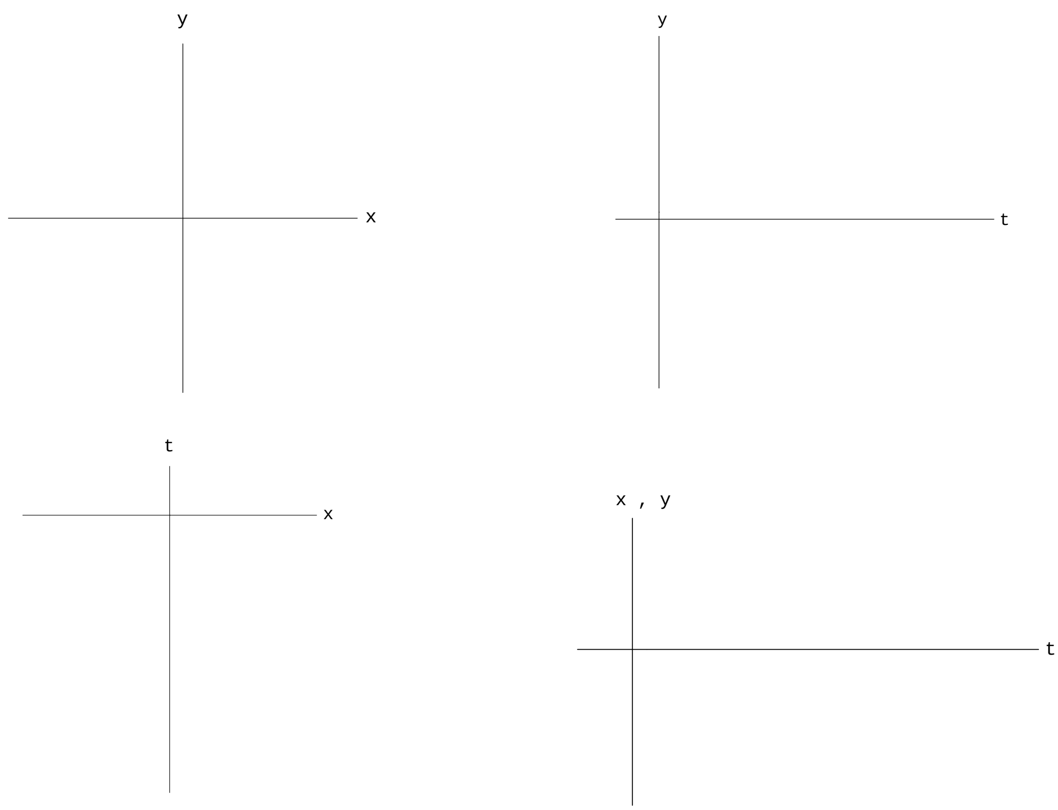
b. Try drawing a semicircle about the origin in the upper half of the *xy*-plane, and observe the $x(t)$ and $y(t)$ graphs that result. Are they what you expected?

c. What do you think will happen on these graphs if you draw the semicircle in the opposite direction?

d. If you draw it twice as fast?

e. Try drawing a spiral from a boundary point to the origin. What sort of physical system might this represent?

2.2 Make, sketch, and explain an experiment of your own using this tool:



This **Phase Plane Drawing** tool is really just a Cartesian-to-parametric illustration—there is no differential equation involved, but it gives a good feeling for the connection between the graphs. In the **Parametric to Cartesian** tool, the phase point is visually linked to the time series coordinates for x and y. A system of differential equations can be solved numerically to get $x(t)$ and $y(t)$, then the xy phase portrait is constructed directly from these.

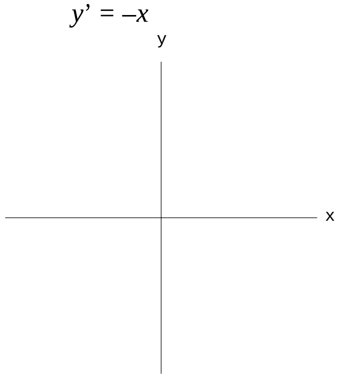
3. Vector Fields for Autonomous Systems

Having established that phase planes are what we often want to look at, we take a moment to observe their construction directly from a system of two differential equations, rather than from the functions $x(t)$ and $y(t)$. It is similar to the construction of a slope field for a one-dimensional differential equation, but there are some differences, such as:

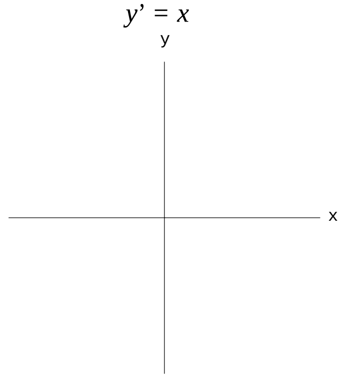
- the vectors may point in any direction, not just left to right as in a slope field
- the curves that flow through a vector field according to the differential equation are called **trajectories,** distinct from the *time series* graphed in the *tx*- and *ty*-planes.

3.1 Use the **Vector Fields** tool to experiment with several different systems. Set enough vectors on the phase plane to visualize the flow of trajectories. Make a quick sketch of a few trajectories for each of the two systems below.

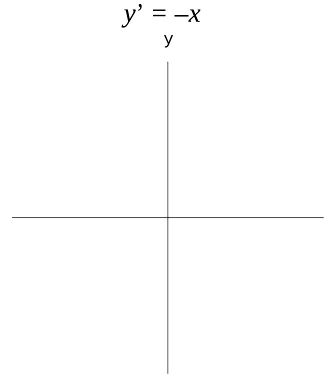
a. $x' = y$
$y' = -x$



b. $x' = -y$
$y' = x$

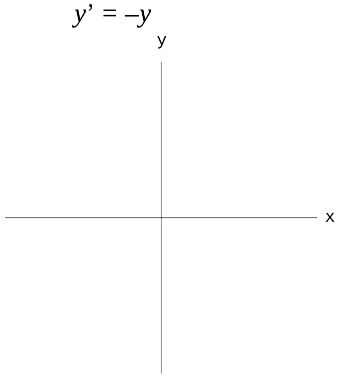


c. $x' = 2y$
$y' = -x$

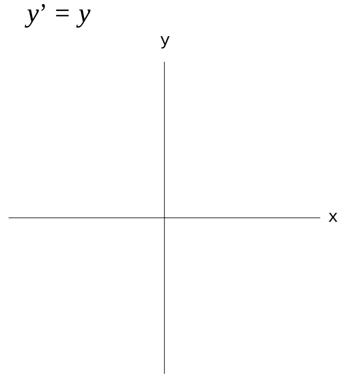


Compare graphs and equations for a, b, and c. How do the differences in the equations explain the differences in the graphs?

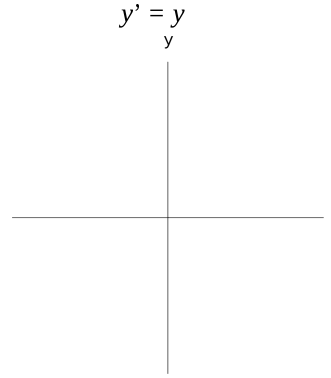
d. $x' = x$
$y' = -y$



e. $x' = x$
$y' = y$

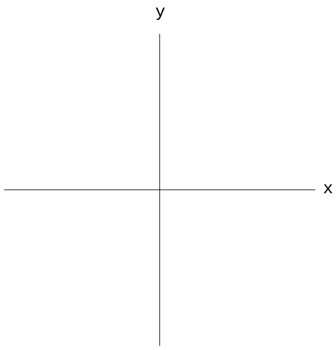


f. $x' = 2x$
$y' = y$

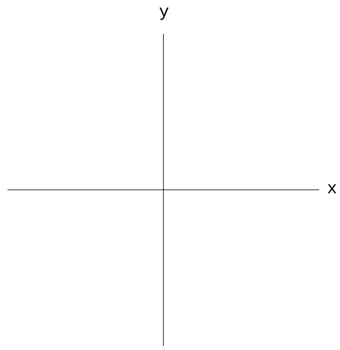


Compare graphs and equations for d, e, and f. How do the differences in the equations explain the differences in the graphs? Look carefully at the ways in which your answers are similar to or different from those in a, b, and c.

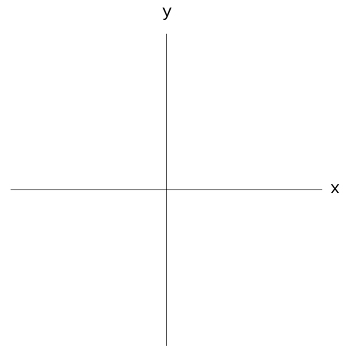
g. $x' = 0$
$y' = y$



h. $x' = y$
$y' = y - x$



i. $x' = x - xy$
$y' = xy/2 - y$



What does an equilibrium look like on a phase plane?

What do the equations look like at an equilibrium?

How many equilibria are there in each of the above graphs, a–i?

3.2 When the time step is 1, the vectors with components *dx/dt* and *dy/dt* are drawn to the correct scale, and their magnitude tells the *speed* of the trajectory.

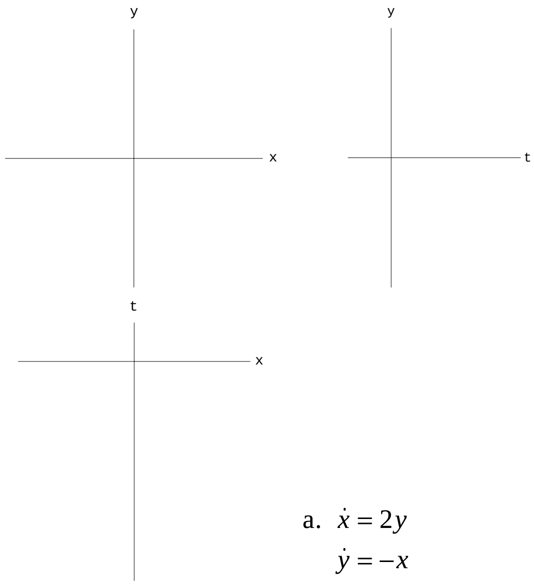
a. It is easy to see that the speed is slowest at the equilibria or fixed points, where $x' = y' = 0$. For the nine equations in Exercise **3.1**, try to find where the speed is maximized for each system and mark the graphs (with a different color).

b. Write a sentence or two explaining why the magnitude of vectors in most vector fields is drawn with constant rather than scaled lengths.

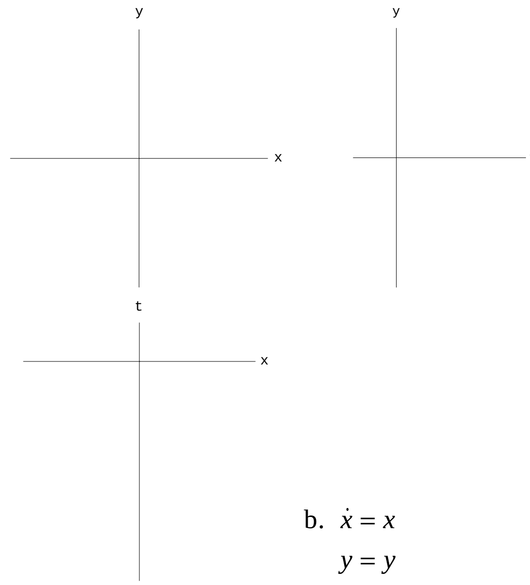
Note: In many software programs, vector fields are drawn without arrowheads. This may not be a great sin if there are arrows drawn on the trajectories, but it is important to realize that in a two-dimensional system, at each point there really is a *vector* with both a magnitude and a direction, as determined by the differential equation.

4. Relating the Graphs

4.1 For each of the equations from Exercise **3.1**, sketch a typical phase plane trajectory from your graph in Exercise **3.1**, and mark a starting point. Then, without the computer, sketch what you would expect the *tx* and *ty* curves to look like for that trajectory. Use the **Two-Dimensional Equations** tool to see if you were right. If you thought of some aspects, but others eluded you, correct your trajectory in a different color and add an explanation. (It is expected that this will happen—it's the explanation of the correction that is important. Show what happened, without erasing.)



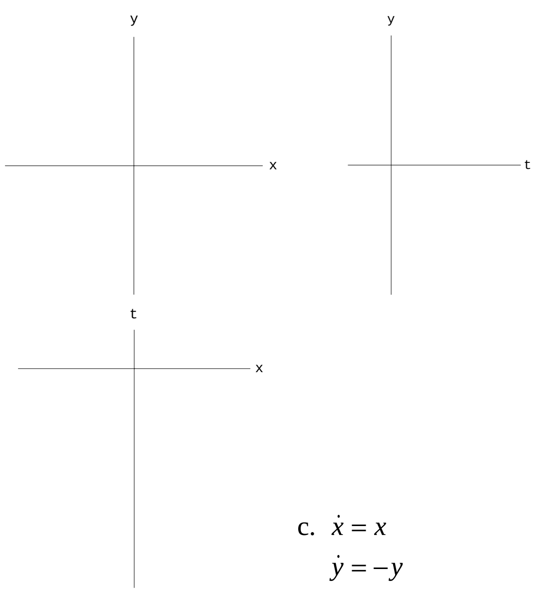
a. $\dot{x} = 2y$
$\dot{y} = -x$



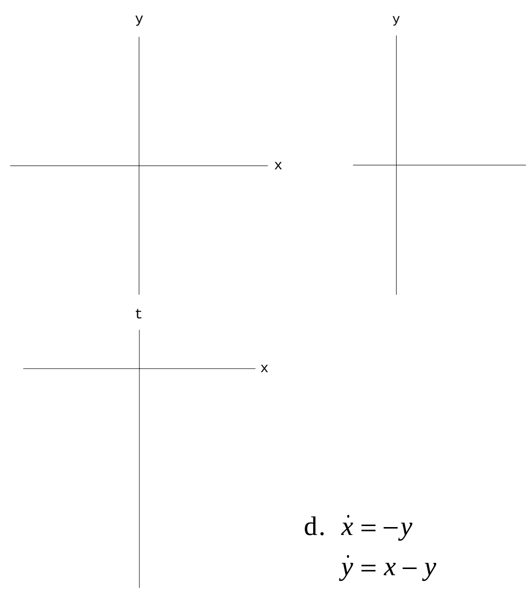
b. $\dot{x} = x$
$y = y$

Explanations of disparities:

Explanations of disparities:



c. $\dot{x} = x$
$\dot{y} = -y$



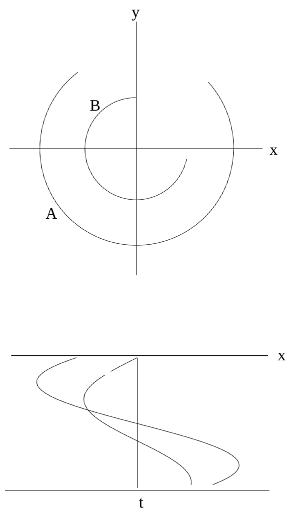
d. $\dot{x} = -y$
$\dot{y} = x - y$

Explanations of disparities:

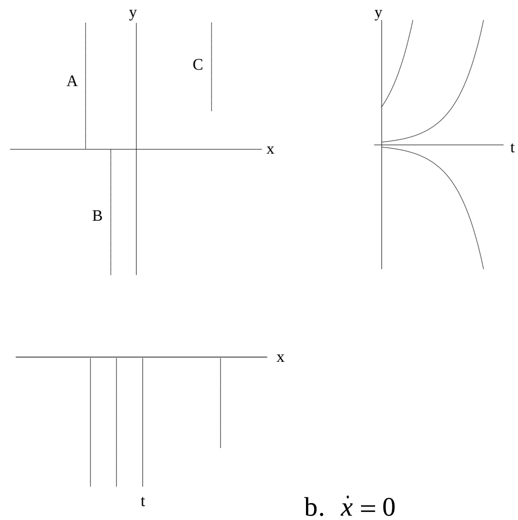
Explanations of disparities:

4.2 Now that you have gained a bit of insight into what to expect for the *tx* and *ty* graphs, look at the following graphs that were drawn simultaneously, and *without the computer*, try to label the *tx* and *ty* curves that match each of the labeled *xy* trajectories.

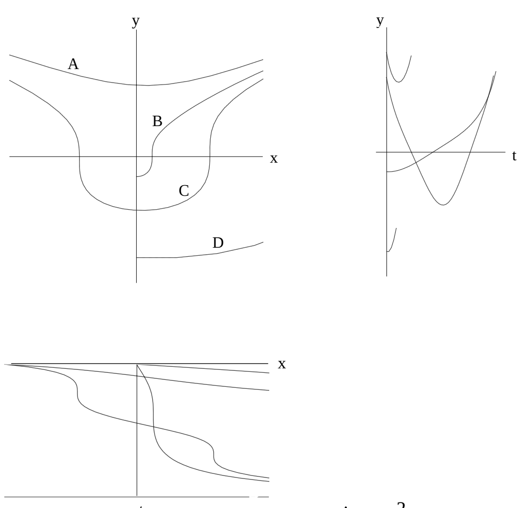
(*Note:* In some cases the *tx* and *ty* curves may have gone a bit longer than shows on the *xy*-plane.) If you get stuck, you could turn to the **Two-Dimensional Equations** tool for help on some of these, but you should still try the next one by hand first. The goal is to become good at this kind of matching, because it comes in very handy.



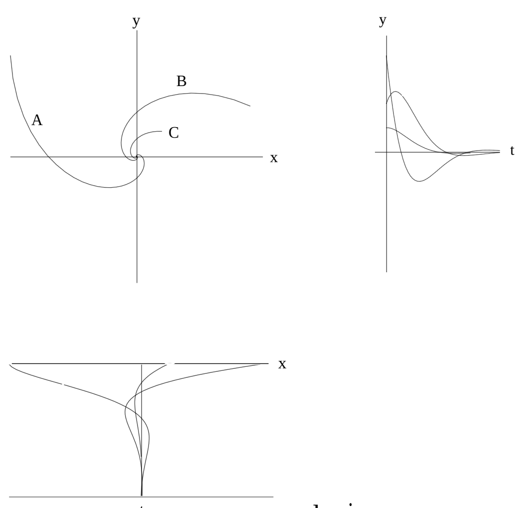
a. $\dot{x} = -y$
$\dot{y} = x$



b. $\dot{x} = 0$
$\dot{y} = y$



c. $\dot{x} = y^2$
$\dot{y} = x$



d. $\dot{x} = -x - y$
$\dot{y} = x - y$

5. Nullclines

The isoclines used to draw slope fields are useful also with vector fields and phase planes, but now that more variables are involved, it is usually easiest just to draw the **nullclines**:

- the isocline of horizontal slopes is where $dy/dt = 0$;
- the isocline of vertical slopes is where $dx/dt = 0$.

5.1 Using a new color, locate the nullclines on each of the phase plane portraits drawn in Exercise **3.1**.

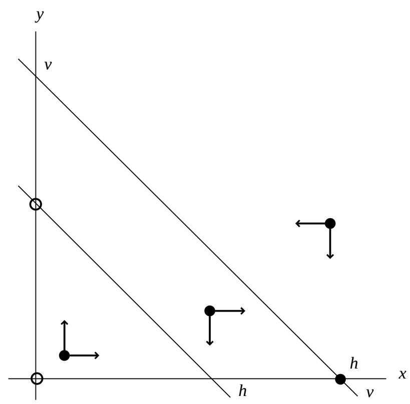
5.2 Equation i in Exercise **3.7** represents an example of a predator-prey system, such as appears in the **Lotka-Volterra** tool from Lab 21, Predator Prey Population Models, with $x = H$ and $y = P$. Open that tool, which focuses on more of the first quadrant. Move the nullclines slightly by moving the sliders. What happens to the overall field and to the equilibria?

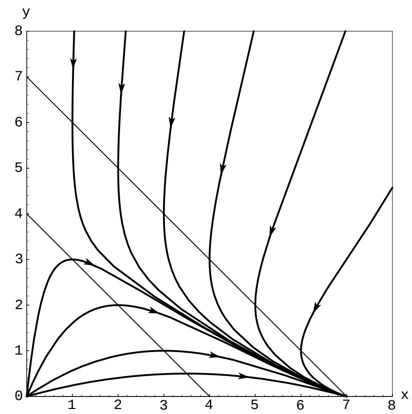
Imagine walking across the vector field from left to right. What happens to the vectors as you cross the isocline of horizontal slopes?

What changes occur in the vectors if you now walk across the vector field from top to bottom, when you cross the isocline of vertical slopes?

While you are in this example, watch a trajectory draw and comment on its speed. Where does it go fast? Where does it go slow? Why?

The regions separated by the nullclines can be qualitatively analyzed (by the differential equations) to tell whether the vectors point right or left, up or down. Equilibria, designated by circles, will occur where an isocline of horizontal slopes meets an isocline of vertical slopes. Here a typical nullcline sketch is given with a phase portrait for a similar example.





5.3 On the phase portrait above, verify that the trajectories pass through the nullclines with proper slopes and confirm that the directions of the trajectories indeed fall in the quadrants predicted on the left-hand nullcline graph.

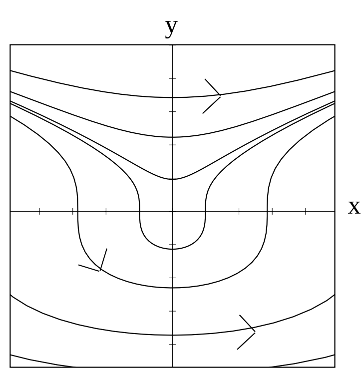
Note: Other examples of this sort can be found in Lab 22, Competing Species Population Models, in the **Competitive Exclusion** tool, where $x = N_1$, $y = N_2$, and (as in the **Lotka-Volterra** tool) the nullclines and vector field can be changed by moving sliders for the various coefficients.

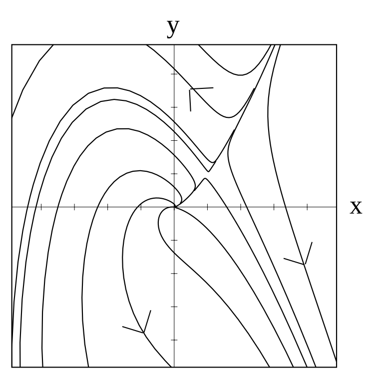
5.4 Naturally, nullclines are not always straight lines. The following equations include excellent examples. With a different color, sketch the nullclines on the following phase portraits, and mark the general left-right and up-down directions in the various regions. Then check your results with either the **Vector Fields** tool or the **Two Dimensional Equations** tool. The last example is a special case—explain what happens.

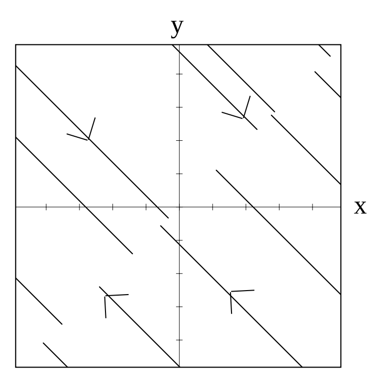
a. $x' = y^2$, $y' = x$

b. $x' = x - y$, $y' = y - x^2$

c. $x' = -x + y$, $y' = x - y$







Lab 17: Tool Instructions

Simple Harmonic Oscillator Tool

Setting Initial Conditions

Click the mouse on the phase plane to set the initial position and the initial velocity, or click the mouse on the time series graph to set the initial position (initial velocity defaults to zero). Clicking while a trajectory is being drawn will stop the trajectory.

Time Series Buttons

The buttons labeled
[ ] **position**
[ ] **velocity**
[ ] **acceleration**
toggle the time series graphs on and off.

Other Buttons

Click the **[Pause]** button to stop a trajectory without canceling it.
Click the **[Continue]** button to resume the motion of a paused trajectory.
Click the mouse on the **[Clear]** button to remove all output from the graphs.

Parametric to Cartesian Tool

Buttons

Click the mouse on the **[Project x]** button to show the corresponding position of *x* on the phase plane.
Click the mouse on the **[Project y]** button to show the corresponding position of *y* on the phase plane.
Click the mouse on the **[Project x and y]** button to show the corresponding positions of both *x* and *y* on the phase plane as phase point coordinates.
Click the **[Pause]** button to stop a trajectory without canceling it.
Click the **[Continue]** button to resume the motion of a paused trajectory.

Phase Plane Drawing Tool

Drawing in the Phase Plane

Press and hold down the mouse button, then move the mouse to draw in the *xy* phase plane.

Buttons

Click the mouse on the **[Clear]** button to remove all output from the graphs.

Vector Fields Tool

Setting Initial Conditions

Click the mouse on the *xy* graphing plane to set the initial conditions for a trajectory or a point for a vector.
Clicking while a trajectory is being drawn will stop the trajectory.

Equations

Click the arrow button to the left of the equations to pop up the list of equations.
Click an equation to select it.

Drawing Mode Buttons

Click the mouse on the **[Vectors]** button to set vectors when you click on the *xy* plane.
Click the mouse on the **[Solutions]** to display a solution curve when you click on the *xy* plane.

Time Step Buttons

Click the mouse on a button in the Δt list to set the time step for vectors and trajectories.

Other Buttons

Click the mouse on the **[Clear]** button to remove all output from the graph planes.
Click the mouse on the **[Draw Field]** button to draw a grid of vectors over the *xy* graphs.

Two Dimensional Equations Tool

Setting initial conditions

Click the mouse on any of the three graphing planes to set the initial conditions for a trajectory.
Clicking while a trajectory is being drawn will stop the trajectory.

Equations

Click the arrow button to the left of the equations to pop up the list of equations.
Click an equation to select it.

Buttons

Click the mouse on the **[Clear]** button to remove all output from the graphs.
Click the mouse on the **[Draw Field]** button to draw a grid of vectors over the *xy* graphs.
Click the mouse on the **[Pause]** button to stop a trajectory without canceling it.
Click the mouse on the **[Continue]** button to resume the motion of the paused trajectory.

Lotka-Volterra Tool

Setting Initial Conditions

The initial conditions for the active trajectory are displayed to the right of the *HP* graphing plane. They are set using either the mouse or the keyboard.

1. Click the mouse on the *HP* graphing plane to set $H(0)$ and $P(0)$.
2. Click the mouse on the value for one of the initial conditions displayed beside the graph to activate a keyboard editor. Set a new number using the number keys, the right and left arrow keys, and the Delete key. Press the **[Return]** key or click the mouse away from the number to leave the editor, set the initial value, and start the trajectory.
3. Clicking while a trajectory is being drawn will stop the trajectory.

Parameter Sliders

Use the sliders to set the growth rate *a*, the predation rate *b*, the predator mortality rate *c*, and the food conversion rate *d*.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the mouse on the **[Clear]** button to remove all output from the graphs.
Click the mouse on the **[Pause]** button to stop a trajectory without canceling it.
Click the mouse on the **[Continue]** button to resume the motion of the paused trajectory.