16 Linear Classification

Tools Used in Lab 16

- Parameter Path Animation
- Parameter Plane Input
- Matrix Element Input
- Four Animation Paths

We have four parameters in the planar system, $\dot{x} = ax + by$ and $\dot{y} = cx + dy$, but to get a parameter plane, we want only two! How do changes in location on the parameter plane affect the phase plane portraits? Take a tour and find out!

1. The Parameter Plane

The planar system of equations can be written

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (1)$$

or equivalently, $\dot{\mathbf{x}} = \mathbf{Ax}$, from which we obtain the second-order differential equation with constant coefficients called the trace, *tr*, and determinant, *det*.

$$\ddot{x} - \underbrace{(a+d)}_{tr\mathbf{A}}\dot{x} + \underbrace{(ad-bc)}_{det\mathbf{A}}x = 0 \quad (2)$$

with the characteristic equation:

$$\lambda^2 - tr\mathbf{A}\lambda + det\mathbf{A} = 0 \quad (3)$$

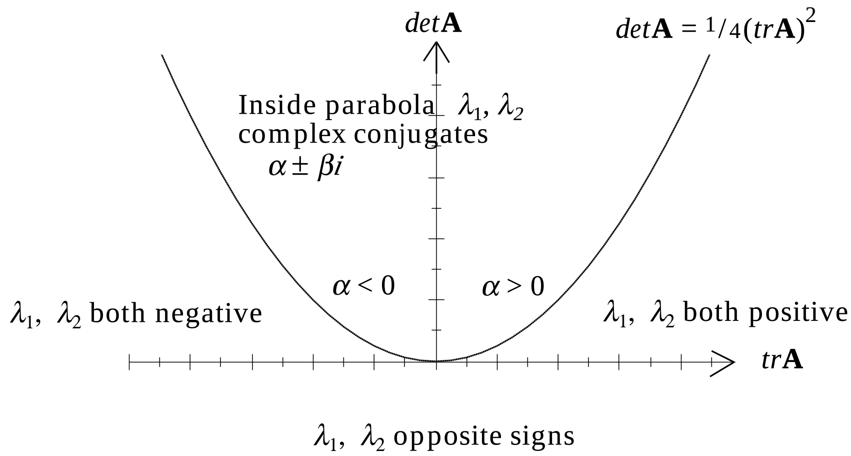
We can get the roots of this equation via the quadratic formula:

$$\lambda = \frac{tr\mathbf{A} \pm \sqrt{(tr\mathbf{A})^2 - 4det\mathbf{A}}}{2} \quad (4)$$

so that the sign of the discriminant, $\Delta = (tr\mathbf{A})^2 - 4det\mathbf{A}$, determines whether there are two real unequal roots $\lambda_1 \neq \lambda_2$, one repeated real root λ_1, or no real roots.

1.1 True or false? Two different matrices cannot have the same trace, determinant, and characteristic equation. Explain if true, or give a counterexample if false.

We construct a **parameter plane** with tr**A** for the horizontal axis and det**A** for the vertical axis. The parabola $\det\mathbf{A} = (tr\mathrm{A})^2/4$ is the locus where the discriminant is 0.



2. Take the Tour

Start the **Parameter Path Animation** tool and begin the tour. Think of riding a railroad car along a path in the parameter plane and watching the phase plane scenery go by. Whenever a small change in a parameter produces a marked change in the behavior of the trajectories, we say a bifurcation has occurred. As you can see here, the bifurcations occur as we move from one colored zone to the next. Several important concepts are illustrated here. We will examine them more carefully using other tools.

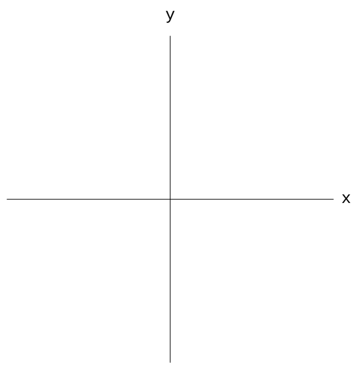
What is a fixed point? The motion of a point along a trajectory in the phase plane can be described by means of $(\dot{x}, \dot{y})$, which tells us how the *x*-coordinate and *y*-coordinate are changing with respect to time. Imagine a point flowing along a trajectory. At any point (*x*, *y*) where $(\dot{x}, \dot{y})$ = (0, 0), we have a fixed point—there is no change. Our imaginary point would also remain at the origin if placed there. You can see that the system is in equilibrium at a fixed point. The question becomes "What kind of equilibrium?"

2.1 Every linear system of the form (1) will have a fixed point at the origin. Can such a system have more than one fixed point? Explain.

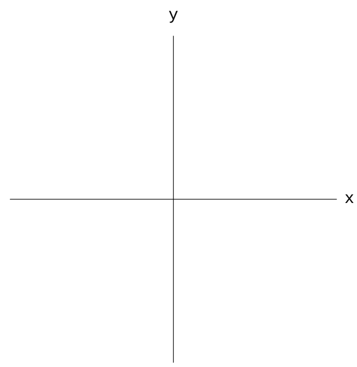
The behaviors of the trajectories in the neighborhood of the fixed point allow us to classify the fixed point. Is it an **attracting** or **repelling** fixed point? In other words, do trajectories in the neighborhood of the fixed point approach it or go away from it as $t \to \infty$? If neither, it is called a **neutral** fixed point. There are many ways of describing fixed points, but to get a better feel for their meanings, do the following exercise.

2.2 Use the **Parameter Plane Input** tool. The origin is always a fixed point. Select a pair of parameters by clicking in one of the colored zones. Look at the resulting values of the λ's, and make a phase portrait by clicking on the initial values for a few trajectories in the phase plane. Make a rough sketch of the portraits below the values for the λ's. Note that each phase portrait is named on the screen. Include those names as labels on your sketches. Label the equilibrium points as attracting, neutral, or repelling. Include only the major colored zones and the positive vertical axis. We will look at the other borderline cases later.

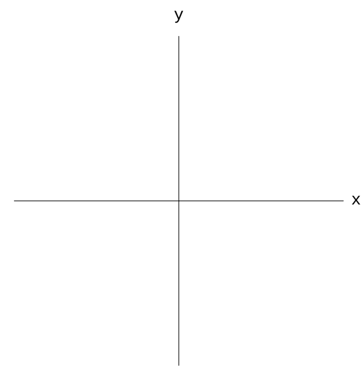
a. $\lambda_1 > \lambda_2 > 0$



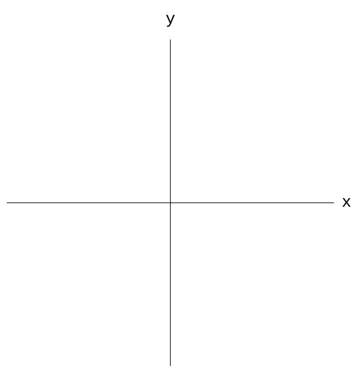
b. $\lambda_1 < \lambda_2 < 0$



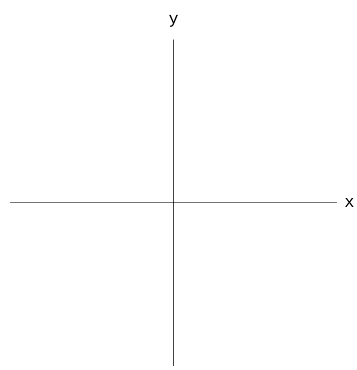
c. $\lambda_1 < 0, \quad \lambda_2 > 0,$
(or vice versa)



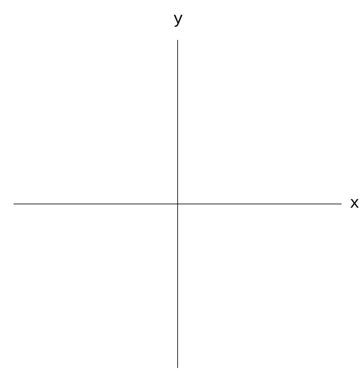
d. $\lambda = \pm\beta i$
$\beta \neq 0$



e. $\lambda = \alpha \pm \beta\, i$
$\alpha > 0, \beta \neq 0$



f. $\lambda = \alpha \pm \beta\, i$
$\alpha < 0, \beta \neq 0$

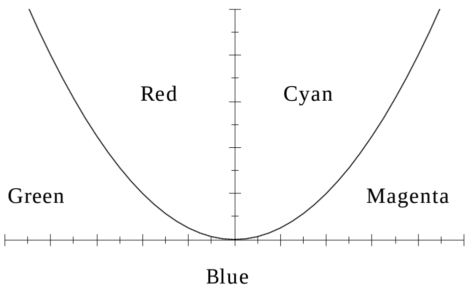


g. Which of these phase portraits has “closed orbits”? This behavior characterizes phase portraits for periodic motion.

h. Which one of these phase portraits represents simple harmonic oscillations?

Transitions

2.3 Use the **Parameter Plane Input** tool to observe the changes in the pictures and the roots, λ, of the characteristic Equation (3) that occur as you move the mouse across boundaries from one colored zone to the next. Set parameter plane points on either side of each boundary and set some points in the *xy*-plane to observe trajectories.



From blue to magenta:

From magenta to cyan (lighter blue-green):

From cyan to white (positive vertical axis):

From white to red:

From red to green:

From green to blue:

3. A Real Example

3.1 The Mass-Spring Problem, from Labs 9 and 10

a. Consider the mass-spring problem with damping, where the mass $m = 1$, the damping constant is δ, and the spring constant $k = 1$. The resulting second-order differential equation is $\ddot{x} + \delta\dot{x} + x = 0$. Write this equation as a planar system of linear differential equations in the standard fashion:

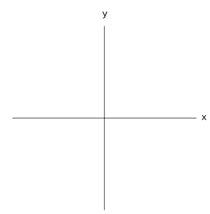
$$\dot{x} = y$$

$$\dot{y} =$$

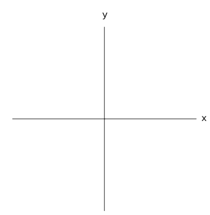
b. What is the corresponding matrix **A** so that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$?

c. Use the **Parameter Plane Input** tool to check the phase portraits for the fixed point as you change the values of matrix element δ from –3 to 3. What is the trace of matrix **A** in terms of δ?

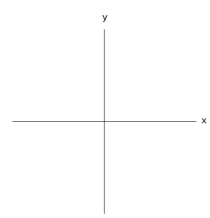
d. What is the color of the zone or border that corresponds to an underdamped oscillator? Sketch the phase portrait.



e. What is the color of the zone or border that corresponds to critical damping? Sketch the phase portrait.



f. What is the color of the zone or border that corresponds to the overdamped case? Sketch the phase portrait.



g. Can you interpret the results for negative values of element δ? *(Hint:* When $\delta < 0,\ tr\mathbf{A} > 0$.)

h. Suppose there is no damping. What kind of fixed point would the origin be?

The matrices used so far have had $\begin{bmatrix}0 & 1\end{bmatrix}$ in the upper row. This choice reflects the fact that a linear second-order differential equation can always be rewritten so that $\dot{x} = y$ is the first equation in the system. The **Matrix Element Input** tool allows you to choose other real matrix entries. We will use this feature to investigate a few more cases.

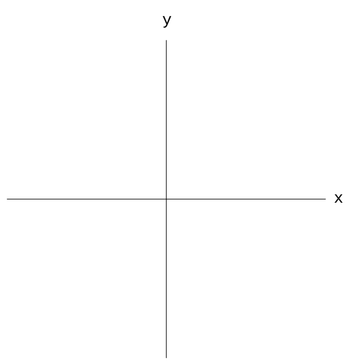
4. Borderline Cases

The most important borderline case divides regions of stable and unstable spirals, sometimes called **spiral sinks** and **spiral sources**, respectively. In the phase plane the corresponding trajectories are ellipses. Recall that **closed orbits** are indicative of periodic motion. The fixed point for this case is a called a **neutral center**; neutral because the trajectories are neither attracted to nor repelled by the fixed point. This situation characterizes frictionless, or **conservative**, systems.

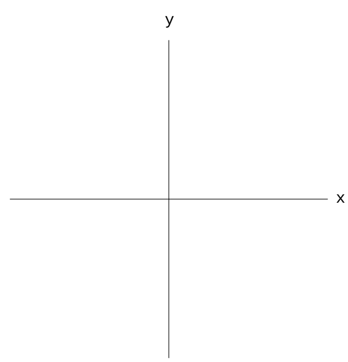
4.1 Use the **Parameter Plane Input** tool to examine the change from spiral to circle to spiral again as you move the cursor across the positive vertical axis. If your touch is exceedingly fine you can set a parameter point that fills the *xy* graph with a slowly decreasing spiral as the orbit just misses being periodic. Note that the trajectory stops when a spiral takes too much time. You can also interrupt a prolonged trajectory with a mouse click.

4.2 Use the tool called **Four Animation Paths** to observe the behavior of the other borderline cases. Sketch phase portraits of typical behaviors for each of the cases along the parabola. Label them with the names on the screen (for example, degenerate node).

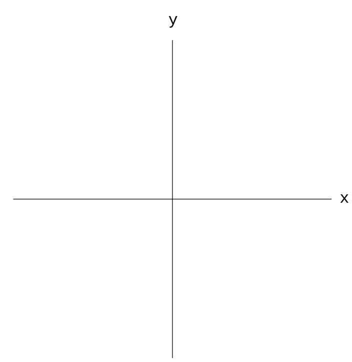
a. $\lambda_1 = \lambda_2 = \lambda > 0$



b. $\lambda_1 = \lambda_2 = \lambda = 0$



c. $\lambda_1 = \lambda_2 = \lambda < 0$



Note that in **(a)** and **(c)** another portrait exists. We need to examine this case more closely. If you have done Lab 15, Linear Algebra, you know this part already. If not, we introduce the basic idea here, in section 5. For more information, refer to Lab 15.

Observe that moving along the horizontal axis, where $det\mathbf{A} = 0$, gives us a line or a plane of **non-isolated fixed points**. You can think of a **degenerate node** as being the borderline case between a spiral and a node, where one tries to deform into the other.

5. The Role of Eigenvectors in Borderline Cases

An **eigenvector** for a matrix **A** is a nonzero vector $\vec{v}$ for which

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where the **eigenvalue,** λ, is a constant. An eigenvector may be stretched, shrunk, reversed, or left alone by the matrix, but it is never rotated. A given eigenvalue has many **eigenvectors.** The eigenvectors for a given λ form a vector space called, not too surprisingly, the **eigenspace.**

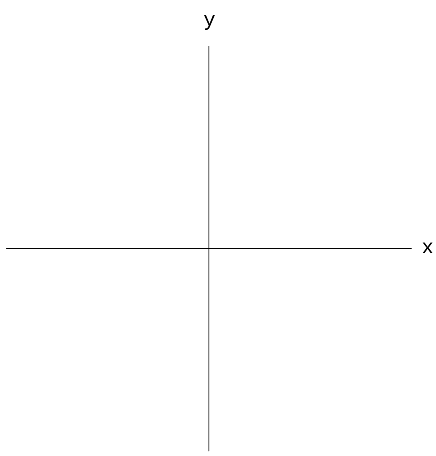
Now look at what happens in the two cases with repeated (nonzero) eigenvalues, $\lambda_1 = \lambda_2 = \lambda \neq 0$.

Case 1. There is only one linearly independent eigenvector. We've seen this case. It is by far the most common. The phase portrait shows that the fixed point is a degenerate node.

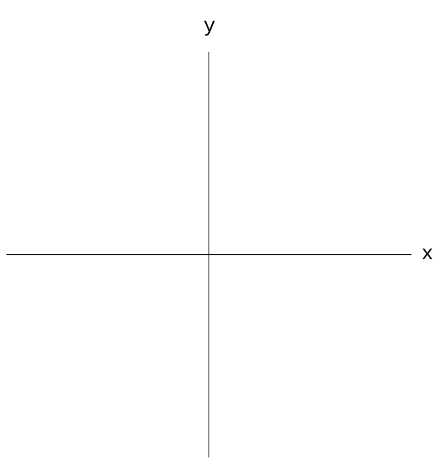
Case 2. There are two linearly independent eigenvectors, and since there is only one two-dimensional vector space containing these eigenvectors, every vector must be an eigenvector. That means that every vector gets stretched or shrunk, etc., in the same way.

5.1 Use the **Matrix Element Input** tool to try the matrix $\mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ for any nonzero λ.

a. Sketch the phase plane portrait. This is called a **star node**. Is it attracting or repelling (that is, is λ positive or negative)? How is this fact related to the sign of λ?



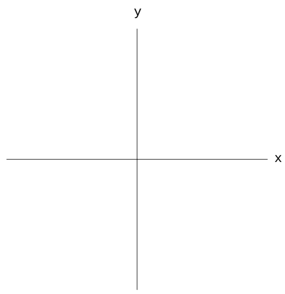
b. Now, using the same λ, try the matrix $\mathbf{A} = \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}$ and vary k away from zero by increments until you get a nice degenerate node. Write your final matrix below, and sketch the phase portrait.



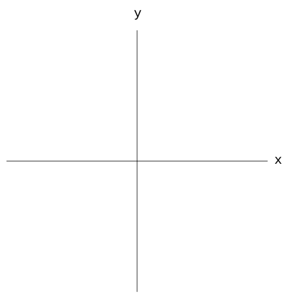
6. Additional Exercises

6.1 Using the **Matrix Element Input** tool to classify the fixed point (or points), find the roots, λ, of the characteristic equation, and sketch the phase portraits for the various matrices **A** where $\dot{\vec{x}} = \mathbf{A}\vec{x}$. (Note that these are the same problems as those in the last part of Lab 15, Linear Algebra.)

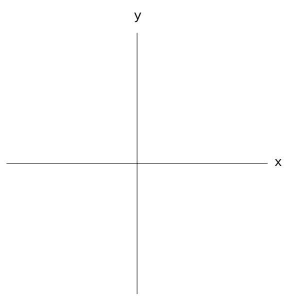
a. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$



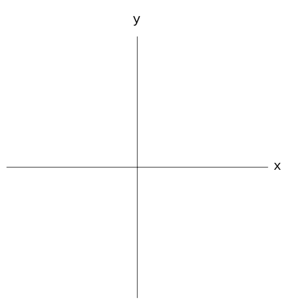
b. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$



c. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



d. $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$



6.2 Show that the characteristic Equation (3) is exactly the same as the characteristic equation obtained from

$$det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$

Lab 16: Tool Instructions

Parameter Path Animation Tool

Buttons

Click on the arrow buttons to control the animation sequence according to the parameter path. Use the double arrow buttons to play the sequence forward and backward. Use the single arrow buttons to advance and reverse the sequence one frame at a time. To stop a play sequence, use a single arrow button.

Parameter Plane Input Tool

Parameter Plane

Move the mouse over the left parameter plane to change the trace and determinant.
Click on the parameter plane to set the determinant and the trace and define the matrix, then click on the *xy* plane to the right to start trajectories. Click on the parameter plane again to release the parameter point and select a new trace and determinant.

Setting Initial Conditions

After setting a point on the parameter plane, click the mouse on the *xy*-graphing plane to set the initial conditions for a trajectory.
Clicking in the *xy*-plane while the trajectory is being drawn will start a new trajectory.

Buttons

Click the mouse on the **[Clear]** button to remove all the trajectories from the *xy*-graph.

Matrix Element Input Tool

Matrix Element Values

Click on a matrix element button in the lower-left corner of the screen to activate the text editor.
Type in new values using the right and left arrow keys, the **[Delete]** key, and the number keys.
Press return or click elsewhere to exit the text editor and set the matrix.

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial conditions for a trajectory.
Clicking while a trajectory is being drawn will stop the trajectory.

Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from the *xy*-graph.
Click the mouse on the **[Draw Field]** button to draw a grid of vectors over the *xy*-graph.

Four Animation Paths Tool

Buttons

Click on the arrow buttons to control the animation sequence of the parameter path. Use the double arrow buttons to play the sequence forward and backward. Use the single arrow buttons to advance and reverse the sequence one frame at a time. To stop a sequence, use the single arrow button.
Click on the path buttons in the lower-left corner of the screen to choose the path on the parameter plane that defines the animation sequence—parabolic, horizontal, vertical, or rectangular.