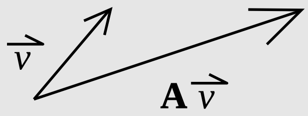


15 Linear Algebra

Tools Used in Lab 15

The Matrix Machine
The Eigen Engine

A matrix **A** *is not just a table of numbers. It is a machine that takes an input vector* $\vec{v}$ *and transforms it into an output vector* $\mathbf{A}\vec{v}$ *where* $\mathbf{A}\vec{v}$ *denotes multiplication of the vector by the matrix. Like an ad for a miracle diet, the "before" and "after" pictures of the vector are usually dramatically different! How does a matrix transform vectors?*

1. Visualizing Matrices as Transformations

The Matrix Machine tool allows you to see how an input vector (yellow) is stretched, rotated, inverted, projected, or otherwise transformed by a matrix to another vector (blue). Try this process for the following matrices.

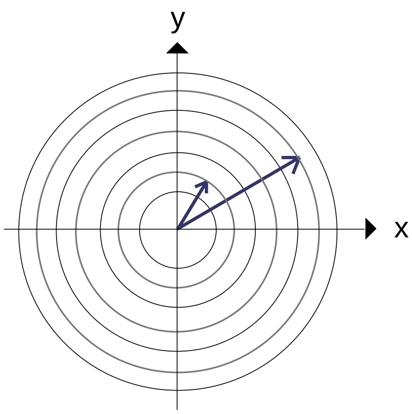
1.1 Input the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ by clicking the arrows to the left or right of each number to lower or raise the value. As you move the mouse around over the *xy*-plane, the mouse coordinates define the yellow input vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then the tool plots the blue output vector $\vec{w} = \mathbf{A}\vec{v} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$. Watch how the output vector $\vec{w}$ changes.

How does the length of $\vec{w}$ compare with the length of $\vec{v}$? *Hint:* Look at the circles.

How is the output vector related to the input vector? What is the geometrical effect of the transformation matrix **A**? Sketch some representative $\vec{v}$, $\mathbf{A}\vec{v}$ pairs.

1.2 Try $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Sketch some representative $\vec{v}$, $\mathbf{A}\vec{v}$ pairs. Drag the mouse around until you find an input vector $\vec{v}$ that gets transformed into $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find this $\vec{v}$ using algebra.

True or false? For any given $\vec{w}$, you can always find a $\vec{v}$ that gets transformed into that $\vec{w}$, and that $\vec{v}$ is unique (that is, there is only one $\vec{v}$ that will work). Explain your reasoning.



1.3 Try $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Sketch some representative $\vec{v}$, $\mathbf{A}\vec{v}$ pairs. By moving the mouse around and watching the output, show that there are many different input vectors that all produce the output vector $\vec{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (These input vectors are said to be in the nullspace, or kernel, of **A**.) Find an equation for all the input vectors $\vec{v}$ in the nullspace.

Now find all the different input vectors that get transformed into $\vec{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. How are these vectors related to the nullspace?

You must have noticed that the output vectors $\vec{v}$ always lie along a certain line. This set of all possible $\vec{v}$ is called the range of **A**. Find a formula for the vectors in the range of **A**.

2. Visualizing Eigenvectors and Eigenvalues

The Eigen Engine tool allows you to see what eigenvalues and eigenvectors mean geometrically. As you input a vector by dragging the mouse around, the tool first converts the vector into a unit vector $\vec{v}$, and then $\vec{v}$ is transformed by multiplying it by the matrix **A**. The resulting output vector $\vec{w} = \mathbf{A}\vec{v}$ is plotted—notice that it usually points in a different direction from $\vec{v}$.

However, for certain special choices of $\vec{v}$, it sometimes happens that $\vec{v}$ and $\vec{w}$ lie along the same line! When this happens, we say that $\vec{v}$ is an **eigenvector,** and the line on which $\vec{v}$ and $\vec{w}$ lie is the **eigen direction**. The screen freezes when you find an eigenvector and displays the **eigenvalue**, the factor by which the unit input vector was stretched or contracted. You can read off the magnitude of the eigenvalue as the radius on the polar grid. The sign of the eigenvalue is positive if $\vec{v}$ and $\vec{w}$ point in the same direction, it is negative if they point in opposite directions, and the eigenvalue is 0 if $\vec{w} = 0$. Click elsewhere on the graph to free up the vectors.

2.1 Find the eigenvectors and eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Show that if $\vec{v}$ is an eigenvector, so is $-\vec{v}$, and they have the same eigenvalue.
Then prove that this is true in general for any matrix **A**.

2.2 Using **The Eigen Engine** tool, find the eigenvectors and eigenvalues of the following matrices. In some cases, you will find eigenvectors lying along two or more separate eigendirections, whereas in others you may find only one or possibly no eigendirections. Sketch the eigenvectors.

a. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

b. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

c. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

d. $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

Lab 15: Tool Instructions

The Matrix Machine Tool

Input Vectors

Move the mouse over the *xy* plane to define the input vector and create the output vector.

Matrix Element Values

Click the pointers to the left and the right of the matrix elements to increase or decrease their values.

The Eigen Engine Tool

Input Vectors

Move the mouse over the *xy* plane to define the input vector and create the output vector. Both the input and the output vectors will freeze when they coincide with an eigenvector. Click the mouse away from the vectors to release them from the eigenvector.

Matrix Element Values

Click the arrows to the left and right of the matrix elements to increase and decrease their values respectively.