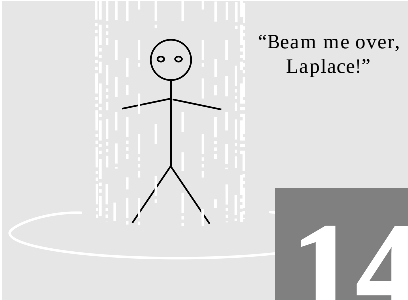


14 Laplace Transforms

Tools Used in Lab 14

- Laplace: Definition
- Laplace: Transformer
- Laplace: Derivative
- Laplace: Vibration and Poles
- Laplace: Translation
- Laplace: Solver
- Laplace: Shifting Theorem
- Laplace: Shift and Step
- Laplace: Delta Function
- Laplace: Convolution Example
- Laplace: Convolution Theorem

Where do linear differential equations act like algebraic equations? In the space of Laplace transforms.

1. Definition and Properties

Definition of the Laplace Transform

For a function f defined on $[0,\infty)$, the Laplace transform is the function of s defined by

$$\mathbf{L}\{f(t)\}(s) \equiv F(s) \equiv \int_0^\infty e^{-st} f(t)\,dt \qquad (1)$$

for all s for which the integral converges. We say that $F(s)$ is now a function in the ***s*-domain**.

1.1 In order to understand the concepts that are part of this definition, open the **Laplace: Definition** tool. This tool illustrates the case when $f(t) = \sin(t)$, but this simple example can lead us to a more general understanding of the definition for any function f. The following questions will lead the way. Begin by considering the upper-half of the screen.

a. For a positive value of s, what does multiplication by e^{-st} do to the function f?

b. What happens to $e^{-st}f(t)$ as you increase the value of s?

c. As you may recall from calculus, the integral (1) with an infinite limit is called an **improper integral**. This integral is defined to be the $\lim_{T\to\infty}\int_0^T e^{-st} f(t)dt$ and is said to **converge** whenever the (finite) limit exists. In the lower-left part of the screen, we look at an intermediate step where we plot the function $I(s,T)=\int_0^T e^{-st} f(t)dt$ versus T. (Note that you can select a value for T by using the cursor on either graphing plane on the left of the screen.) That is, we evaluate the definite integral from 0 to T. Describe in words what this integral represents.

d. For a given value of s, what happens to $I(s,T)$ as T becomes large? Does that happen eventually for every value of $s > 0$? Can you visually determine the $\lim_{T\to\infty}\int_0^T e^{-st} f(t)dt$ for each value of s?

e. Now put the cursor on the s-slider and slide s from 0 to the end. Describe what happens to all three graphs as s increases.

i. the graph of $e^{-st}f(t)$ versus t:

ii the graph of $I(s,T)$ versus t:

iii. the graph of $F(s)$ versus s:

1.2 What kinds of functions on $[0,\infty)$ have Laplace transforms? Not every function does, but we can guarantee the existence of a Laplace transform for any function f on $[0,\infty)$ that is piecewise continuous (so it can be integrated) and is of **exponential order** α. The latter condition means that the function f must not grow too fast as t increases, in that $|f(t)|$ must be bounded eventually by some exponential function $Me^{\alpha t}$ for some $M > 0$. Note that it is precisely this property (that $f(t) = \sin(t)$ was "dampable" by multiplication by e^{-st} for some $s > 0$) that plays such a significant role in the **Laplace: Definition** tool.

a. Not every function has an exponential order. Can you think of one that does not have that property?

Also note that there may be functions out there with neither exponential order nor piecewise continuity that have Laplace transforms. Later, we'll meet the Dirac delta function which has neither property and, in fact, is not strictly a function, yet it has a useful Laplace transform.

1.3 Open up the **Laplace: Transformer** tool. Send a few standard functions through the "transformer" by clicking on the arrow to see what they transform to in the s-domain. Note that every function in the list meets the two criteria in the preceeding paragraph. What do you observe about the transformed function $F(s)$ as s becomes large?

Because Laplace transforms are just integrals, they share the property with derivatives and integrals of being linear operators. This means that the **linearity property** $\mathbf{L}\{a\,f + b\,g\} = a\mathbf{L}\{f\} + b\mathbf{L}\{g\}$ holds for any functions f and g whose Laplace transforms exist in a common domain, and any constants a and b.

The fundamental reasons that Laplace transforms are so useful are linearity and the way they act on derivatives. These properties allow us to change linear differential equations with initial conditions (initial-value problems in the t-domain) into algebraic equations in the s-domain. Then we can solve them in the s-domain and transform them back to obtain the solutions in the t-domain.

The Laplace Transform of a Derivative

Differentiation in the t-domain is multiplication by s in the s-domain (for zero initial conditions).

The Derivative Theorem

If the functions f and f' are continuous on $[0,\infty)$ and f'' is piecewise continuous on $[0,\infty)$, and if f, f' and f'' have exponential order α, then for $s > \alpha$,

$$\begin{aligned} \mathbf{L}\{f'\}(s) &= s\,F(s) - f(0) \\ \mathbf{L}\{f''\}(s) &= s^2F(s) - sf(0) - f'(0) \end{aligned} \tag{2}$$

where $f(0)$ and $f'(0)$ are the values of f and f' corresponding to $t = 0$.

1.4 Open the **Laplace: Derivative** tool to investigate this property for $f(t) = \cos(2t)$. What is $f(0)$? What is $\mathbf{L}\{f'(t)\}$ according to Equation (2)? Is that what you would get if you took the derivative of $\cos(2t)$ first and then took the Laplace transform? Show this algebraically.

1.5 Illustrate the Derivative Theorem for the function $f(t) = t^2 + 2$ without using the computer.

1.6 Use Equation (2) to transform the linear initial-value problem $x'' + x = e^{-t}$, with initial conditions $x(0) = 1, x'(0) = 0$, to an algebraic equation. Consult the **Laplace: Transformer** tool for the transforms of the forcing function. Solve for $X(s)$, the Laplace transform of the solution $x(t)$. You need not go further at this point.

In order to finish solving the above initial value problem, we must make certain that we know how to transform $X(s)$ in the s-domain to the unique solution $x(t)$ in the t-domain. There is a bit of a problem. Because the integral in definition (1) works for piecewise continuous functions, there may be many functions with the same Laplace transform. If there is a *continuous* function with a given transform, *there can be only one!* It is this function that we choose to be the unique inverse transform. If there is no such function we can still recover a function from its transform, but there may be ambiguities at the points of discontinuity.

With these reservations, we say that the **inverse Laplace transform** of a function $F(s)$ is the function f of t that satisfies $\mathbf{L}\{f\}(s) = F(s)$. We denote it by $\mathbf{L}^{-1}\{F(s)\}$.

1.7 When you operated the **Laplace: Transformer** tool in Exercise **1.6**, did you notice that it is also an "inverse transformer"? Try the inverse transformer on the following two transforms to find the unique continuous function $f(t)$ that corresponds to each $F(s)$. Write the functions below. You will need to use the linearity property. More importantly, you will need to rewrite each term to fit the formulas provided by the **Laplace: Transformer** tool.

a. $\mathbf{L}^{-1}\{\frac{4}{s^3} - \frac{2}{s^2+1}\}$

b. $\mathbf{L}^{-1}\{\frac{1}{s-2} + \frac{5}{s} - \frac{2s}{s^2+9}\}$

1.8 Find the inverse transform for the function $X(s)$ determined in Exercise **1.6** in order to completely solve the given initial-value problem $x'' + x = e^{-t}$ where $x(0) = 1$, $x'(0) = 0$. Note that it is standard to use the method of partial fractions here.

Consider now the simplified unforced mass-spring problem modeled by the differential equation $x'' + 2bx' + x = 0$ with initial conditions $x(0) = 1$, $x'(0) = 0$. This physical situation was illustrated in the **Damped Vibrations** tool for Lab 10. Note that we can transform the initial-value problem using Equation (2):

Wait! Look at the denominator! Except for the label for the variable (s instead of λ), it is exactly the left-hand side of the characteristic equation $\lambda^2 + 2b\lambda + 1 = 0$ that we obtained when we solved the original initial value problem in Lab 10. Recall that the characteristic equation gave us the eigenvalue λ. The zeros of the denominator of the Laplace transform of the solution to an initial-value problem are called the *poles* of the transform. The significant fact is that the poles of the transform carry the same information as the eigenvalues of the characteristic equation.

1.9 To get these concepts clearly in mind, open the **Laplace: Vibration and Poles** tool and vary the damping constant b to obtain the four possible cases ($b = 0$, $0 < b < b_c$, $b = b_c$, $b > b_c$). Describe what happens to the poles as b is increased from 0 through 2.

1.10 Now look at the lower graph of $X(s)$ vs. s. As you move the b slider notice that the poles show up as asymptotes for the overdamped and critically damped cases. Why don't they show up for the underdamped case?

This situation is our first hint of the deeper structure of the s-domain. In more advanced courses, you will learn to treat s as a complex variable and $X(s)$ as a complex-valued function of a complex variable. However for the remainder of this lab, we will stay with the introductory treatment and consider s and $X(s)$ as real.

2. Forcing Functions

Let's look at some new forcing functions. The first theorem deals with translation along the s-axis. To get a feel for this theorem, open the **Laplace: Translation** tool, and use the slider to vary parameter a. Find both transforms and inverse transforms. This theorem is also called the First Shift Theorem.

The Translation Theorem

If the Laplace transform of f exists for $s > \alpha$, so that $\mathbf{L}\{f(t)\}(s) = F(s)$, then

$$\mathbf{L}\{e^{at} f(t)\}(s) = F(s-a) \quad \text{for } s > \alpha + a.$$

2.1 At this point you are ready to see the overall picture of how Laplace transforms work to solve initial-value problems. Open the **Laplace: Solver** tool and send through a few forcing functions. Sketch the solutions for the following initial-value problems with initial-values $x(0) = 1,\ x'(0) = 0$.

a. $x'' + x = e^{-t}$

b. $x'' + 3x' + 2x = \sin(t)$

As yet it is difficult to see the advantages in using Laplace transforms. We'll begin to appreciate them more as we start looking at special functions. The first one of interest is the **unit step function:**

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

2.2 Use the equation $\mathbf{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$ to obtain the Laplace transforms of the following functions. First sketch the functions, and then write them in terms of unit step functions. Then take the transforms.

a. $f(t) = \begin{cases} 0 & t < 1 \\ 2 & 1 < t < 3 \\ -1 & 3 < t < 4 \\ 0 & t > 4 \end{cases}$

b. the square wave; $f(t) = \begin{cases} 1 & 2n < t < 2n+1 \\ -1 & 2n+1 < t < 2n+2 \end{cases}$ where n is an integer.

c. Express $g(t) = (1 - u(t-1))e^{-t}$ in the form of the functions in parts a. and b. and sketch a graph of the function. Then find the Laplace transform.

How do we take the Laplace transform of the product of a unit step function with another function? We now need a new theorem. This one is sometimes called the Delayed Function Theorem or, if you are counting, the Second Shift Theorem. We call will it the Shifting Theorem.

The Shifting Theorem

Suppose $F(s) = \mathbf{L}\{f\}(s)$ exists for $s > \alpha \geq 0$. If a is a positive constant,

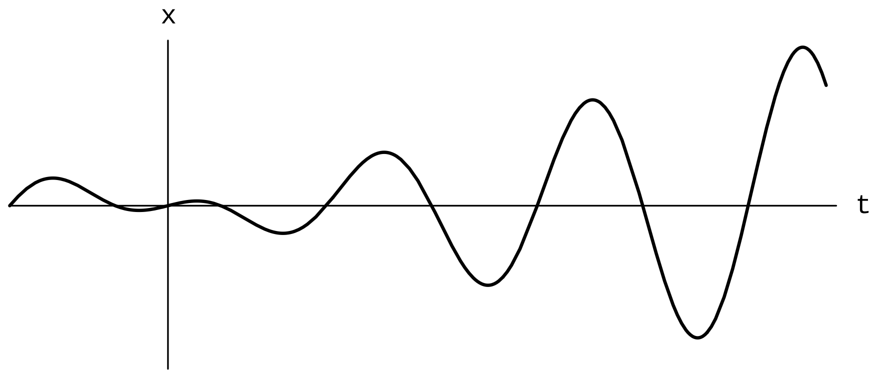
$\mathbf{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$.

An alternate form that is particularly useful is

$\mathbf{L}\{u(t-a)\, f(t)\} = e^{-as}\mathbf{L}\{f(t+a)\}$.

2.3 Open the **Laplace: Shifting Theorem** tool. Try a few values of parameter a on the slider. Take both transforms and inverse transforms by activating the arrows with the cursor. Where do “delayed functions” come in? Open the **Laplace: Shift and Step** tool while you consider this question. Then complete the following exercises.

a. You may need to review what a shifted function is. For instance, if we replace the variable f in $f(t)$ by $t - a$ where $a < 0$, the graph of the function will be shifted to the right a units. Check this result for the sine function in the tool and apply this idea to a new function $f(t) = \frac{t}{4}\cos(2t)$. The graph of this function before shifting is given below. Sketch the shifted function.



b. Describe the changes in the graph when a function $f(t)$ is multiplied by $u(t - a)$.

Explain and make a sketch using the function $f(t) = \frac{t}{4}\cos(2t)$.

c. Describe the new graph obtained for $u(t - a)f(t - a)$. Did we turn on the function at $t = a$ instead of $t = 0$? In other words, did we "delay" the function? Explain and make a sketch of the delayed function for $f(t) = \frac{t}{4}\cos(2t)$.

2.4 Suppose you found a function of s in the s-domain that was an e^{-as} multiplied by the Laplace transform of a known function. Would you suspect a delayed function? Try a few examples below by finding the inverse Laplace transform.

a. $\mathbf{L}^{-1}\{e^{-2s}\frac{5}{s^3}\}$

b. $\mathbf{L}^{-1}\{e^{-(\pi/2)s}\frac{s}{s^2+9}\}$

The most interesting special function is the **Dirac delta function** (which mathematicians call a **generalized function** because we have to "stretch" our concept of what a function is in order to include it). We are going to view the delta function in terms of a limiting process. First, however, let's consider the idea from physics of an **impulsive force,** a force applied over some brief time interval. Then, to calculate the **impulse** we must integrate over the time interval**.**

$$\text{Impulse} = \int_b^a f(t)\,dt$$

We define a new function: $f_h(t) = \frac{1}{h}(1 - u(t-h)) = \begin{cases} 1/h & 0 < t < h \\ 0 & t > h \end{cases}$

2.5 Determine the impulse for f_h for the time interval $[0, \infty)$.

Look at the tool **Laplace: Delta Function**. Try both sliders. Note that as $h \to 0$, each "box" continues to have unit area. We are going to define the Dirac delta function to be the limit of the f_h's as $h \to \infty$ so that the property of having area 1 under the curve is maintained. For this reason, the delta function $\delta(t)$ is often referred to as the **unit impulse function**.

Are you having trouble believing in the delta function? What is the problem (that is, aside from the fact that the delta function is infinitely tall, infinitely thin, and has an area under the curve of 1)? If you are willing to accept it, you will find it very useful in modeling a variety of physical problems, especially those involving impulsive forces and voltage surges. However, it is not your everyday function. Until the development of distribution theory, some mathematicians viewed it with suspicion.

2.6 Open up the **Laplace: Solver** tool again and try some delta functions and unit step functions.

a. Sketch the solution for $x'' + 3x' + 2x = u(t-2)$.

b. Sketch the solution for $x'' + 3x' + 2x = \delta(t-2)$.

3. The Convolution Theorem and the Transfer Function

How are the solutions to a linear differential equation related to the forcing function? The answer is surprisingly easy to see in the *s*-domain. However, in the *t*-domain we need a new idea—convolution!

The **zero initial state**, where all initial conditions are zero, will be assumed for the rest of this lab. We are getting very close to the main idea!

3.1 What happens if we use $\delta(t)$, the unit impulse function at $t = 0$, as our forcing function? (Recall that $L\{\delta(t)\} = 1$.) Consider the general second-order linear differential equation with constant coefficients:

$ax'' + bx' + cx = \delta(t)$

For the zero initial state, show that $X(s) = \dfrac{1}{a\,s^2 + b\,s + c}$. (3)

The right-hand side of Equation (3) is called the **transfer function G(s)**, and it represents the response of the system to the unit impulse function. It carries information that we have seen in another form. *Notice that the zeros of the denominator of* $G(s)$ *are precisely the roots of the characteristic equation* $a\,\lambda^2 + b\,\lambda + c = 0$. Now we can recognize the zeros of the denominator as the poles of the transfer function G(s).

Many times the transfer function is exactly what you need to investigate a mechanical or electrical system. To find the solution in the *s*-domain for any forcing function for which a Laplace transform $F(s)$ exists, you need only take the product $G(s)\,F(s)$. Of course, if you need the solution in the *t*-domain, you need to find $\mathbf{L}^{-1}\{G(s)F(s)\}$. That, unfortunately, is harder than it looks!

3.2 Give an example to show that $\mathbf{L}^{-1}\{G(s)F(s)\}$ does *not* equal $g(t)f(t)$.

What then is $\mathbf{L}^{-1}\{F(s)G(s)\}$? The answer is given by the Convolution Theorem, but first we must define the convolution of two functions on f and g defined on $[0,\infty)$.

The convolution of two piecewise continuous functions f and g on $[0,\infty)$ is defined to be

$$(f * g)(t) \equiv \int_0^t f(T)g(t-T)\,dT, \text{ or equivalently,}$$

$$(f * g)(t) \equiv \int_0^t f(t-T)g(T)\,dT.$$

In order to get a feel for the convolution as the integral of the product of $f(T)$ and $g(t-T)$, open the tool **Laplace: Convolution Example.** The title of this tool is a little misleading because Laplace transforms do not play a role, as yet. The very instructive example illustrated here is due to S. Farlow and can be found in his text *An Introduction to Differential Equations and their Applications*, McGraw-Hill, Inc., 1994, p. 304.

3.3 Open the **Laplace: Convolution Theorem** tool. Select functions and activate the convolve arrow followed by the $\mathbf{L}$-arrow; then activate the $\mathbf{L}$-arrow followed by the multiply arrow. What do you observe about the result?

This next theorem should not surprise you!

The Convolution Theorem

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$ and of exponential order α, then

$\mathbf{L}\{f * g\}(s) = F(s)G(s)$, or equivalently,

$\mathbf{L}^{-1}\{F(s)G(s)\} = f * g$ where $F(s) = \mathbf{L}\{f\}(s)$ and $G(s) = \mathbf{L}\{g\}(s)$.

Now we can fill in some of the missing ideas in the **Laplace: Solver** tool. The desired inverse transform of the solution in the *s*-domain is the convolution of $g(t)$ and $f(t)$, where $g(t) = \mathbf{L}^{-1}\{G(s)\}(t)$ for the transfer function $G(s)$ and where $f(t)$ is the forcing function, so that $\mathbf{L}^{-1}\{G(s)F(s)\}(t) = (g * f)(t) = x(t)$.

Lab 14: Tool Instructions

Laplace: Definition Tool

Parameter Sliders

Use the slider to change the value constant for *s* values between 0 and 4.

Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Laplace: Transformer Tool

Parameter Sliders

Use the slider to set the value constant for *b* between 0 and 4.

Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Drawing Mode Buttons

Click the mouse on the **[L]** button to see the graphical output of the Laplace Transform of a selected function.

Click the mouse on the **[Inverse]** button to change the **[L]** button to the **[L^{-1}]** button.

Click the mouse on the **[L^{-1}]** button to see the graphical output of the inverse Laplace Transform of a selected function.

Click the mouse on the **[Laplace]** button to change the **[L^{-1}]** button to the **[L]** button.

Other Buttons

Click the mouse on the **[Set Function]** button to choose a function.

Click on a function to select it.

Laplace: Derivative Tool

Drawing Mode Buttons

Click the mouse on the **[L]**, **[Differentiate]**, and / or **[Derivative Theorem]** buttons to see graphical output.

Other Buttons

Click the mouse on the **[Clear]** button to remove all output from the graphs.

Laplace: Vibrations and Poles Tool

Parameter Sliders

Use the slider to change the values for the parameter *b*.

Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Drawing Mode Buttons

Click the mouse on the **[Start]** button to see graphical output.

Other Buttons

Click the **[Pause]** button to stop a trajectory without canceling it.

Click the **[Continue]** button to resume the motion of a paused trajectory.

Laplace: Translation Tool

Parameter Sliders

Use the slider to set the value constant for *a* values between –1 and 4.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Drawing Mode Buttons

Click the mouse on the **[L]** or the **[L^{-1}]** buttons to toggle between a function's graphical output and the graphical output of its Laplace Transform.

Laplace: Solver Tool

Drawing Mode Buttons

Click the mouse on the **[L]**, **[Solve]**, and **[L^{-1}]** arrow buttons to see algebraic and graphical output.

Other Buttons

Click the mouse on the function buttons below to choose a function, its initial conditions, and the type of damping.

Laplace: Shifting Theorem Tool

Parameter Sliders

Use the slider to set the value constant for *a* values between 1 and 4.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Drawing Mode Buttons

Click the mouse on the **[L]** button to see graphical output.

Laplace: Shift and Step Tool

Parameter Sliders

Use the slider to set the value constant for *a* values between 0 and 4.
Press the mouse down on the slider knob and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Laplace: Delta Function Tool

Parameter Sliders

Use the slider to set the value constants for *a* values between 0 and 3, and for *h* values between 0 and 4.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Laplace: Convolution Example Tool

Parameter Sliders

Use the slider to set the value constant for T values between 0 and 2.

Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Laplace: Convolution Theorem Tool

Drawing Mode Buttons

Click the mouse on the **[L]**, **[convolve]**, and / or **[multiply]** arrow buttons to see graphical output.

Other Buttons

Click the mouse on the **[Clear]** button to remove all output from the graphing plane.