



13

Electrical Circuits

Tools Used in Lab 13

Series Circuits
Damped Vibrations: Energy
Van der Pol Circuit

A series circuit with an inductor, resistor, and capacitor can be represented by $L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t)$, a second-order linear differential equation with constant coefficients. Look familiar?

1. Circuit Laws

We know the following two facts about our series circuit (due to Kirchhoff):

1. The current in every part of the circuit is the same.
2. The sum of the voltage drops around the circuit must be equal to the input voltage $V(t)$.

From these facts, we can obtain a differential equation where each term on the left represents a voltage drop (in volts) across the designated circuit element and each constant L , R , and $1/C$, can be viewed as a constant of proportionality in henries, ohms, or (farads)⁻¹, respectively. Note that q represents the charge (on the capacitor) as a function of time t and $I = \frac{dq}{dt}$ represents current in the circuit as a function of time t . The equation can have three forms, based on convenience. The **basic series circuit equation** is

$$L\dot{I} + RI + \frac{1}{C}q = V(t) \quad (1)$$

In terms of charge, $q(t)$, in coulombs (using the fact that $I = \frac{dq}{dt}$):

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t) \quad (2)$$

In terms of current, $I(t)$, in amperes (by differentiating every term of the basic equation):

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}(t) \quad (3)$$

For the most part, we use Equation (2) in terms of $q(t)$. Note that it looks surprisingly like the equation for the spring:

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

In fact, within the linear operation of the circuit elements, the analogy is very tight and quite useful in understanding the behavior of circuits.

2. L-C Circuits

2.1 The L-C Circuit: $L\ddot{q} + \frac{1}{C}q = 0$

Open the **Series Circuits** tool. Set $R = 0$ and $A = 0$ *precisely*. Without an input voltage, the only energy in the circuit is due to the initial conditions. Start with (nontrivial) initial conditions by clicking the mouse on the phase plane to set $q(0) = q_0$ and $q'(0) \equiv I(0) = I_0$. Are you surprised to get the equivalent of simple harmonic motion on a spring?

a. Solve the linear differential equation for $q(t)$ above in terms of L and C .

b. What is the natural frequency of oscillation for the circuit? Compare it to the natural frequency of the spring, $\omega_0 = \sqrt{\frac{k}{m}}$.

The capacitor stores electrical energy and has a voltage drop in proportion to its charge, so that $1/C$ is *analogous to k , the spring constant*. The inductor stores magnetic energy and builds a ‘back’ voltage in proportion to the change in the current flowing through it, dI/dt , so *the inductance L is analogous to the mass m* , (i.e. the inertia or resistance to change in velocity) in the spring system.

Notation: for convenience, we use “ \approx ” for “is analogous to,” so that

$$L \approx m$$

$$\frac{1}{C} \approx k$$

2.2 Energy in the L-C Circuit

Look at the energy graph on the **Damped Vibrations: Energy** tool. Use the analogy with the mass-spring system to obtain the expression of the magnetic (kinetic) energy in the inductor and the electric (potential) energy in the capacitor. Notice that without resistance there is no dissipation of energy in the circuit, so the total energy remains constant. Remember that $q \approx x$, the displacement, and $I \approx v$, the velocity \dot{x} . Complete the sentence for the energy in the circuit:

mass-spring $E_{total} = E_{kinetic} + E_{potential} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

L-C circuit $E_{total} = E_{magnetic} + E_{electric} = \underline{\hspace{2cm}}$

2.3 The Forced L-C Circuit: $L\ddot{q} + \frac{1}{C}q = A\cos(\omega t)$

Use the **Series Circuits** tool to determine the response when the frequency of the input voltage $V(t) = A\cos(\omega t)$ is near or equal to the natural oscillating frequency ω_0 of the circuit. Make a sketch of the resulting oscillations when ω is close to but unequal to ω_0 .



3. L-R-C Circuits

3.1 The L-R-C Circuit with $V(t) \equiv 0$: $L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$

- What does the resistance, R , in a series circuit correspond to in the spring system?
- Now set $A = 0$ and set R to a small nonzero value on the sliding scale. What kind of motion do you observe? Give a rough sketch below. State your initial conditions for the charge q_0 and current $\dot{q}(0) = I(0)$.



- Are the charge q and current I underdamped, overdamped, or critically damped?
- Using the analogy with the spring for the critical damping, $b_{cr} = \sqrt{4mk}$, find the value of the resistance R_{cr} that gives critical damping.
- Now use $L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$, via the characteristic equation $L\lambda^2 + R\lambda + \frac{1}{C} = 0$, to find again the resistance R_{cr} that gives critical damping.
- We purposely left the sliders without units so that any consistent set of realistic units could be used. Set $L = 2$ henries and $C = 0.5$ microfarads. Vary the resistance to find the resistance that gives critical damping. With the units we are using, the slider for the resistance is in kilo-ohms (that is, 10^3 ohms) and the time scale is in milliseconds (10^{-3} seconds). Is the critical resistance you discovered the same as the calculated resistance R_{cr} ? If not, figure out the problem. It should be the same. What is the resistance?
- For every value of $R \neq 0$, what is the long-term behavior of $q(t)$ and $I(t)$?

3.2 Energy in the L-R-C Circuit

Now that we have resistance in the circuit, there is heat loss. In fact, the loss due to heat dissipated in the resistor is $E_{dissipated} = E_{total} - (E_{magnetic} + E_{electric})$. Keep the analogy with the mass-spring system firmly in mind. Describe carefully what happens over the long run to the available energy $E_{magnetic} + E_{electric}$ (or in the case of the spring, $E_{potential} + E_{kinetic}$). What happens to the energy that is dissipated?

3.3 The L-R-C Circuit with $V(t) \equiv A \cos(\omega t) : L\ddot{q} + R\dot{q} + \frac{1}{C}q = A \cos(\omega t)$

Reopen the **Series Circuits** tool. Set R to some nonzero values and experiment with the response of the circuit to various values of ω and $A \neq 0$.

- a. Describe what happens on the phase plane and on the time series as t becomes large.

- b. The transient solutions for the charge (and current) are analogous to the transient solutions for the position (and velocity) in Lab 11, Forced Vibrations: An Introduction and die out as time becomes large. Was that your experience in Exercise 3.1(g)?

- c. The steady-state solution depends on the impressed voltage as well as the circuit elements and gives the long-term behavior of the system. **Set $A = 2$ and $\omega = 1$. Set $C = 0.5$ microfarads $R = 1$ ohm, and $L = 1$ henry.** Now try several values for L , $1/C$, and ω . Note that the wiggle at the beginning of the solution curve is due to the transient behavior of the circuit. What is the angular frequency of the steady-state solution?

- d. The transient solution corresponds to the _____ solution (homogeneous or particular?) of the differential equation for the circuit with applied voltage.

The steady state solution corresponds to the _____ solution (homogeneous or particular?).

4. Additional Exercises

4.1 A Simplified Induction Motor

Suppose we have a new induction motor circuit with a problem. The AC power supply overheated at small frequencies, so we added a capacitor of 1 microfarad in series, but now the motor circuit has undesirable resonances as we accelerate it through its natural frequency. There are already 500 ohms of resistance in the circuit, but it has been suggested that an additional resistor be placed in series to damp out the resonance. If the inductance L is 1 henry, find the least resistance required to damp out the resonance in the circuit.

4.2 A Nonlinear Example

Just for fun, open the **Van der Pol Circuit** tool . The resistor is nonlinear and supplies variable damping, that is, positive damping (which dissipates energy) for part of the time and negative damping (which adds energy) for part of the time.

Look for a limit cycle! (That is, try a variety of initial conditions. What happens?) You don't have to look very hard, because the limit cycle is a "global attractor." Describe what happens.

Lab 13: Tool Instructions

Series Circuits Tool

Setting Initial Conditions

Click the mouse on the $q\dot{q}$ plane or the time series graph to set the initial conditions for a trajectory.

Clicking in the $q\dot{q}$ plane or the time series graph while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the sliders to change the values for the parameters L , R , $1/C$, A , and ω . The circuit is automatically redrawn when the slider values are set to zero.

Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled

charge

current

toggle the time series graphs on and off.

Other Buttons

Click the [**Pause**] button to stop a trajectory without canceling it.

Click the [**Continue**] button to resume the motion of a paused trajectory.

Click the mouse on the [**Clear**] button to remove all the trajectories from the graph.

Damped Vibrations: Energy Tool

Setting Initial Conditions

Click the [**Start**] button to start a trajectory using preset initial conditions.

Clicking in the time series graph will set an initial value of x and start a trajectory.

Clicking in the time series graph while a trajectory is being drawn will start a new trajectory.

Parameter Slider

Use the slider to set the damping coefficient, b .

Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled

energy

velocity

Position

toggle the time series graphs on and off.

Other Buttons

Click the [**Pause**] button to stop a trajectory without canceling it.

Click the [**Continue**] button to resume the motion of a paused trajectory.

Van der Pol Circuit Tool

Setting Initial Conditions

Click the mouse on the *iv* graphing plane to set the initial conditions for a trajectory.
Clicking in the *iv* plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the slider to set the nonlinear resistance parameter ϵ .
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled

capacitor voltage

current

nonlinear resistance

toggle the time series graphs on and off.

Other Buttons

Click the mouse on the **[Clear]** button to remove all the trajectories from the graph.