12

Forced Vibrations: Advanced Topics

Tools Used in Lab 12

- Vibrations: Amplitude Response
- Vibrations: Phase Response
- Vibrations: Input/Output

How does a damped harmonic oscillator behave when it is driven by a sinusoidal force? The answer is relevant to the structural vibrations of bridges and skyscrapers in a gusty wind, the tuning of a radio, a car driving over a bumpy road, and many other systems subjected to periodic excitations.

1. Forced Damped Oscillator Response

The second-order linear equation

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F_0\cos\omega t \qquad (1)$$

governs the motion of a damped harmonic oscillator driven by a sinusoidal force. As before in Lab 10, **Free Vibrations**, we suppose that a weight of mass m is attached to a spring of stiffness k and damped by a viscous frictional force of strength c. The variable $x(t)$ describes the displacement of the mass. The new feature is that the system is driven by the periodic force $F_0\cos\omega t$, where F_0 is the driving strength and ω is the driving frequency.

In Lab 10, we saw that if there is no forcing ($F_0 = 0$), the motion eventually damps out: $x(t)\to 0$ as $t\to\infty$. But if $F_0 \neq 0$, the behavior becomes much more interesting, as you'll see below. The applied forcing can counteract the effects of damping. In particular, if the system is forced at frequencies close to its natural frequency, and if the damping is not too large, then "resonance" occurs, thereby causing the oscillations in $x(t)$ to grow to large amplitudes. But what happens if the system is jiggled much faster than its natural frequency, or much slower? The goal of this lab is to help you understand the different types of responses—both resonant and non-resonant—that the system can exhibit.

Equation (1) has five parameters: m, k, c, F_0, ω. This is a lot of parameters to vary. To keep things simple, we assume that $m = 1$, $k = 1$, and $F_0 = 1$. This might seem like an overly special case, but in fact it is completely general, in the following sense: if m, k, and F_0 are all nonzero, we can always convert Equation (1) to an equation of the form

$$\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+x=\cos\omega t \tag{2}$$

by rescaling time and x appropriately. (For now, take that on faith. If you don't believe that anything so wonderful could possibly be true, see the last question at the end of this lab.) The upshot is that we can get away with studying Equation (2), which has only two parameters instead of five, and yet we aren't sacrificing any generality!

1.1 Show that Equation (2) has a particular solution (known as the **forced response**) given by

$$x_p(t)=A\cos(\omega t-\phi) \tag{3}$$

where the **amplification factor** A is given by

$$A=\frac{1}{\sqrt{(1-\omega^2)^2+4b^2\omega^2}} \tag{4}$$

and the **phase lag** ϕ is

$$\phi=\tan^{-1}\left(\frac{2b\omega}{1-\omega^2}\right). \tag{5}$$

2. Amplitude Response

Our goal in this part is to understand the meaning of Equation (4).

Open the **Vibrations: Amplitude Response** tool. The schematic shows a mass on a spring, along with a dashpot that damps the oscillations of the mass. The whole system is being driven by a piston that moves the top of the spring up and down. (More intuitively, imagine holding the top of the spring in your hand, and jiggling it up and down periodically.) The time series graph shows $x(t)$, the solution of Equation (2), for the initial conditions $x(0) = 0$, $\dot{x}(0)=0$, and for the values of b and ω chosen on the sliders. In the upper-right part of the screen, the curve (4) is plotted as a function of ω, for a given value of b. This graph of A vs. ω is known as the **amplitude response curve**.

2.1 Set $b = 0.25$. Then move the slider for ω back and forth, and notice that a horizontal line is being drawn on the time series graph at a height given by A. Now choose $\omega = 1$. Click on the **[Start]** button to see the resulting solution $x(t)$. How is the eventual amplitude of the oscillations in $x(t)$ related to A ? Explain your observations, using the concepts of homogeneous and particular solutions.

2.2 For $b = 0.25$ and $\omega = 1$, what is the value of A predicted by (4)? Does this agree with what you observe in $x(t)$?

2.3 Increase the driving frequency to $\omega = 2$. How does $x(t)$ change, compared to the previous results?

So far you have held the damping strength b fixed at $b = 0.25$. Now explore what happens when you vary b.

2.4 As you decrease b below 0.25, describe what happens to the shape of the amplitude response curve. In particular, what happens to the height of the peak in the curve? Also, roughly estimate the value of ω at which the peak occurs.

2.5 Now suppose you increase b. What happens to the height and location of the peak in the curve?

2.6 Perhaps you have noticed that the peak in the amplitude response curve occurs near $\omega = 1$ when b is small, but shifts to the left for larger b. By maximizing Equation (4) with respect to ω, find a formula for the value of ω at which the peak occurs.

2.7 Find the height of the peak, as a function of b. This quantity is known as Q, the **quality factor** of the system. Give an approximate formula for Q if b is small.

Conclusion

We have seen that if the damping b is small, the amplification factor A becomes large when $\omega \approx 1$. This is the phenomenon of **resonance:** if a vibrating system is weakly damped, it exhibits a large amplification factor when driven at a frequency near its natural frequency (which is $\omega = 1$ because of the way we have scaled the parameters). The strength of the resonance is commonly expressed in terms of the peak amplification factor, that is, the quality factor Q of the system. For instance, electrical engineers try to design radio amplifiers with a high Q, to allow for precise tuning and strong amplification of faint signals.

3. Phase Response

In this part, we explore the phase relationship between the oscillating mass and piston. Do they move together, or in opposition, or what? How does the answer depend on ω and b?

3.1 Open the **Vibrations: Amplitude Response** tool again. Set $b = 0.15$. When $\omega = 2$, does the mass eventually move **in phase** with the piston (do they move up together, and down together), or does the mass move in **antiphase** (up when the piston moves down, and vice versa)?

3.2 Repeat the previous question, for $\omega = 0.67$. After the system settles down, does the mass move in phase with the piston, or in antiphase?

3.3 Give a rule of thumb that generalizes the previous results. For $b = 0.15$, when do antiphase oscillations occur, and when do in-phase oscillations occur?

Now open the **Vibrations: Phase Response** tool to explore these issues in more detail. The graph of ϕ vs. ω is called the **phase response curve**. It is a graph of Equation (5) as a function of ω, for a given value of b. The meaning of phase lag ϕ becomes clear if you look at the time series of the forcing $F(t) = \cos \omega t$, and compare it to the response $x(t)$, which eventually approaches $x_p(t) = A\cos(\omega t - \phi)$ after transients decay. The predicted phase lag ϕ is indicated by the green arrow. When $\phi \approx \pi$, the two curves are about half a cycle apart, and hence are in antiphase, whereas when $\phi \approx 0$, they are in phase.

3.4 The **Vibrations: Phase Response** tool includes sliders for both ω and b. Play with the b slider to see how it affects the phase response curve. How does the shape of the curve change as b is varied?

3.5 Using Equation (5), what are the predicted values of ϕ as $\omega \to 0$, $\omega \to 1$, and $\omega \to \infty$?

4. Input vs. Output

Another way to visualize the effects of the periodic drive is to plot the solution $x(t)$ vs. the applied forcing $F(t)$. Such a graph is called an **input-output graph**. The idea is that the input $F(t)$ is converted or "filtered" by the system to produce the output $x(t)$. For instance, in electrical circuits (Lab 13), $F(t)$ might represent an applied AC voltage, and $x(t)$ would represent the output AC current. The shape of the resulting input-output graph provides a great deal of information about the amplitude and phase response of the system.

Open the **Vibrations: Input/Output** tool. As before, Equation (2) is integrated starting from $x(0) = 0$, $\dot{x}(0) = 0$ and for the values of b and ω chosen on the sliders. The left panel shows the input-output graph in the (F, x) plane. The transient part of the solution is shown in gray, and the long-term forced response is shown in red. The right panel shows the time series of the input $F(t) = \cos \omega t$, along with the time series of the output $x(t)$.

Behavior Near Resonance

Throughout this part, set $\omega = 1$.

4.1 Choose a value of b and look at the input-output graph. Prove that the curve approaches an ellipse, and find the Cartesian equations of this limiting ellipse.

4.2 Click on the **[Start]** button to plot the corresponding time series of $x(t)$. Observe that $x(t)$ crosses through 0 at the precise instant when $F(t)$ is a maximum or a minimum. Explain this mathematically.

4.3 By estimating from the graph, what is the height of the ellipse when $b = 0.25$? $b = 0.50$? $b = 1$? Guess a formula for the height of the ellipse as a function of b, and prove this formula if you can. Explain how all this is related to the quality factor Q discussed earlier.

Weak Damping

Throughout this part, set $b = 0.25$.

4.4 What happens to the ellipse for $\omega > 1$? $\omega < 1$?

4.5 What do you think happens to the ellipse in the limit $\omega \to 0$? $\omega \to \infty$? Try to justify your guesses mathematically.

4.6 Set $\omega = 0.25$. Click on the **[Start]** button, wait for the times series to be drawn, and then click on the **[Continue]** button. Explain why the two time series are almost the same. And why is the ellipse confined almost entirely to the 45° diagonal line?

5. Scaling the System

5.1 Show that if m, k, and F_0 are all nonzero, Equation (1) can be rescaled to Equation (2) by introducing new dimensionless definitions of x and t, and defining b appropriately.

Lab 12: Tool Instructions

Vibrations: Amplitude Response Tool

Setting Initial Conditions

Click the **[Start]** button to start a trajectory using preset initial conditions.
Clicking in the time series will set an initial value of x and start a trajectory.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the slider to change the values for the parameters b and ω.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the **[Pause]** button to stop a trajectory without canceling it.
Click the **[Continue]** button to resume the motion of a paused trajectory.

Vibrations: Phase Response Tool

Setting Initial Conditions

Click the **[Start]** button to start a trajectory using preset initial conditions.
Clicking in the time series will set an initial value of x and start a trajectory.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the slider to change the values for the parameters b and ω.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the **[Pause]** button to stop a trajectory without canceling it.
Click the **[Continue]** button to resume the motion of a paused trajectory.

Vibrations: Input/Output Tool

Setting Initial Conditions

Click the **[Start]** button to start a trajectory using preset initial conditions.
Clicking in the time series will set an initial value of $x(t)$ and start a trajectory.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Sliders

Use the slider to change the values for the parameters b and ω.
Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the **[Pause]** button to stop a trajectory without canceling it.
Click the **[Continue]** button to resume the motion of a paused trajectory.