



# Forced Vibrations: An Introduction

Tools Used in Lab 11  
Damped Forced Vibrations

*The addition of a sinusoidal forcing function to the model of a linear oscillator brings a new level of complexity. Transient responses may appear and fade and the long term system response is closely tied to the frequency of the forcing function.*

## 1. The Damped Oscillator with Forcing

In this lab we explore the effect of a sinusoidal forcing function on the damped and undamped mass-spring system. This lab presents an overview. For a more detailed exploration, look at Lab 10, Free Vibrations and Lab 12, Forced Vibrations: Advanced Topics, and their corresponding tools.

The equation for the damped linear oscillator with forcing function  $F_0 \cos(\omega t)$ , where  $F_0$  and  $\omega$  are constants, is given by

$$m\ddot{x} + b\dot{x} + cx = F_0 \cos(\omega t) \quad (1)$$

where  $b > 0$  and  $F_0, \omega \neq 0$ .

Open the **Damped Forced Vibrations** tool. Note that the vertical position as a point on the time series graph corresponds to the vertical position of the mass. You can set the initial position  $x_0$  and velocity  $v_0$  on the phase plane.

- 1.1** Set  $m = 1.5$ ,  $b = 0.5$ ,  $k = 1$ , and  $F_0 = 2$ ,  $\omega = 1$ . Start the graph by setting the initial conditions on the phase plane. Sketch the result below with the three kinds of curves that make up the  $x$  vs.  $t$  graph carefully labeled and distinguished by colored pencil or dotted and dashed lines.



The next few exercises are to help you understand what is going on in this graph.

The position function  $x(t) = x_{transient} + x_{steady\ state}$  is the solution of Equation (1), where  $x_{transient} = x_h$  is the solution to the associated homogeneous equation and  $x_{steady\ state} = x_p$  is the particular solution.

The **transient solution** is the solution of the associated homogeneous equation  $m\ddot{x} + b\dot{x} + kx = 0$ . In the previous lab, we learned that solving the characteristic equation  $m\lambda^2 + b\lambda + k = 0$  using the quadratic formula gives rise to three distinct situations:

- the **overdamped case** when  $b^2 - 4mk > 0$ :

$$x_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \text{ where } \lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \text{ are both negative.}$$

- the **critically damped case** when  $b^2 - 4mk = 0$ :

$$x_h = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \text{ where } \lambda = -\frac{b}{2m}$$

- the **underdamped case** when  $b^2 - 4mk < 0$ :

$$x_h = e^{-\alpha t} (c_1 \cos(\beta t) + c_2 \sin(\beta t)) \text{ where } \lambda_1, \lambda_2 = \alpha \pm \beta i, \alpha = -\frac{b}{2m}, \beta = \frac{\sqrt{4mk - b^2}}{2m}$$

These solutions are called **transient** because each solution tends toward zero as  $t$  gets large.

**1.2** How do the above expressions guarantee that the homogeneous solutions  $x_h$  die out with increasing time? Give a concise explanation for each of the three cases. Note that the critically damped case may require l'Hôpital's Rule.

a. overdamped:

b. critically damped:

c. underdamped:

- 1.3 a. In Exercise 1.1, did the transient solution correspond to the overdamped, critically damped, or underdamped case?
- b. Now vary the slider on the damping constant  $b$ , holding  $m$  and  $k$  constant, to obtain each type of transient behavior. State your values for  $k$  and  $m$  and give the corresponding values of  $b$  for each type of behavior.

### The Steady-State Solution

The first thing to observe is that the frequency of the steady-state solution is exactly that of the external force. Note that the method of undetermined coefficients can be used to find the steady-state solution. Start by varying the frequency of the forcing function to see how the frequency of the steady-state solution is affected. Then vary the amplitude  $F_0$  of the forcing function. Note that the scale on the vertical axis is changing as you change  $\omega$ .

- 1.4 a. True or false: The steady-state solution has the same amplitude as the driving force.
- b. True or false: The steady-state solution may be out of phase with the driving force. (*Hint:* Look at the relative motion of the orange piston and the yellow weight.)

## 2. The Forced Undamped System

Set  $b = 0$  for the remainder of this lab. Equation (1) then becomes:

$$m\ddot{x} + kx = F_0 \cos(\omega t) \quad (2)$$

The position function  $x(t)$  is still the sum of the homogeneous and the particular solutions, but they are no longer designated as transient or steady-state. In fact the homogeneous solution is no longer transient in that it doesn't die out with increasing time and the particular solution may no longer be at all "steady." The particular solution can oscillate or grow in amplitude as time increases, depending on the frequency of the forcing function.

Recall that the homogeneous solution is  $x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$  where the natural frequency of the spring is  $\omega_0 = \sqrt{k/m}$ . Notice that  $x_h$  does not die out with time.

- 2.1. Assume  $\omega \neq \omega_0$ . Note that we can find  $x_p$ , the particular solution for Equation (2), by using the method of undetermined coefficients. As your textbook will show,  $x_h$  is a sinusoidal function of frequency  $\omega_0$ , and  $x_p$  is a sinusoidal function of frequency  $\omega$ . Use the software to observe that the position function  $x$  is the sum of these two sinusoidal functions of different frequencies. Using three colors or dotted and dashed lines, sketch and clearly label the graphs of these functions below.



- 2.2 Now set  $m = 1$ ,  $k = 1$ , and  $\omega = 0.5$  rad/sec. Let the values of  $\omega$  approach  $\omega_0$ . For what values of  $\omega$  do beats begin to appear? What happens as  $\omega$  gets closer and closer to  $\omega_0$ ? Describe what happens to the beats. Do they have a greater amplitude? Is their frequency increasing or decreasing? Remember that the scale on the vertical axis is changing as you change  $\omega$ .

- 2.3 Now we get to the exciting part! Set  $\omega = \omega_0 = 1$ . The effects you observe are called **resonance**.

Rather than expanding the scale to accommodate vibrations of huge amplitude, the screen shows the consequences of a “real” resonance effect. To see what happened mathematically, we must solve Equation (2) using the method of undetermined coefficients. Now  $x_p = A t \cos(\omega_0 t) + B t \sin(\omega_0 t)$  for some appropriate  $A$  and  $B$ . As your textbook will show, this process yields the solution

$x_p = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$ . Sketch a graph of the particular solution below. What happens to the amplitude of the oscillations as time increases?



## Lab 11: Tool Instructions

### Damped Forced Vibrations Tool

#### Setting Initial Conditions

Click the **[Start]** button to start a trajectory using preset initial conditions.

Clicking in the time series will set an initial value of  $x$  and start a trajectory.

Clicking in the plane while a trajectory is being drawn will start a new trajectory.

#### Parameter Sliders

Use the slider to change the values for the parameters  $m$ ,  $b$ ,  $k$ ,  $F$ , and  $\omega$ .

Press the mouse down on the slider knob for the parameter you want to change and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

#### Time Series Buttons

The buttons labeled

**transient**

**steady state**

**position**

toggle the time series graphs on and off.

#### Other Buttons

Click the **[Pause]** button to stop a trajectory without canceling it.

Click the **[Continue]** button to resume the motion of a paused trajectory.

