10 Free Vibrations

Tools Used in Lab 10
- Damped Vibrations
- Critical Damping

A mechanic pushes down on the front end of a car to observe the vibrations. If they don't damp out quickly the shock absorbers need to be replaced. A certain amount of friction is necessary for a smooth ride. How can we find the right amount for the most effective damping?

1. Simplifying the System

The second-order linear equation

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0 \qquad (1)$$

governs the motion of a weight of mass m attached to a spring of stiffness k, damped by a viscous frictional force of strength c. The variable $x(t)$ describes the mass's displacement from its equilibrium position.

Our first goal is to visualize the behavior of this system for different values of the parameters m, k, and c. With three separate parameters to vary, you might worry that there could be an enormous number of different possibilities to consider, but miraculously, a *single* combination of these parameters tells you essentially how the system will behave—that's the point of the following question.

1.1 Show that if we define a new time variable $\tau=\omega_0 t$, where $\omega_0=\sqrt{k/m}$ is the natural frequency of the undamped system, then Equation (1) simplifies to $\frac{d^2x}{d\tau^2}+2b\frac{dx}{d\tau}+x=0$. Give a formula for $2b$ in terms of m, k, and c. (The factor of 2 in front of b probably looks strange, but we'll see later that it simplifies the formula for the eigenvalues.)

The advantage of the new variables is that we have only one parameter to vary, namely b. In other words, we can set $m = 1$ and $k = 1$ without loss of generality! Therefore, from now on we can restrict our attention to the simpler equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + x = 0\,. \tag{2}$$

Note that we have also reverted to the familiar letter t instead of τ.

2. Eigenvalues

2.1 Show that Equation (2) has a solution of the form $x = e^{\lambda t}$, and find and solve the characteristic equation for the eigenvalue λ in terms of b. Show that the eigenvalues are real and negative for $b > 1$ (*overdamped* case), and complex with Re $\lambda < 0$ for $0 < b < 1$ (*underdamped* case). Show that there is a repeated eigenvalue for $b = 1$ (*critically damped* case). What is special about the eigenvalues when $b = 0$ (*undamped* case)?

Open the **Damped Vibrations** tool. Its purpose is to help you gain some intuition about the meaning of undamped, underdamped, critically damped, and overdamped vibrations. The slider allows you to set the parameter b.

2.2 Move the slider all the way to the left, so that $b = 0$. This is the familiar case of a simple harmonic oscillator without damping. The eigenvalues are $\lambda = \pm i$, as shown in the display. These eigenvalues are also plotted as dots in the top panel, which shows their location in the complex λ plane.

Describe how the eigenvalues change as you drag the slider to the right.

2.3 As you drag the slider, notice that the eigenvalues move around in the complex λ plane. The picture suggests that the eigenvalues lie on the unit circle $|\lambda| = 1$ for the underdamped case $0 < b < 1$. Prove this.

3. Time Series and Animations

Besides showing the eigenvalues as a function of b, the **Damped Vibrations** tool also shows the solution $x(t)$ of Equation (2), starting from an initial condition $x(0) = 1$, $\dot{x}(0) = 0$. The corresponding motion of the mass is shown in the animation alongside the graph of $x(t)$. Notice that the animation and the time series have the same vertical scale: the current position of the mass is shown as a moving yellow dot on the graph of $x(t)$. The gray curves show the "envelope"—the time series stays between these curves.

3.1 Find the solution $x(t)$ for the given initial conditions for the underdamped case.

3.2 Move the slider toward $b = 1$. How do the vibrations change as b approaches 1 from below?

3.3 What is the qualitative difference between the solutions for $0 \le b < 1$ and $b \ge 1$?

4. What's Special about Critical Damping?

Open the **Critical Damping** tool. It allows you to compare the solutions of Equation (2) for different values of b. All the solutions start from the same initial condition, $x(0) = 1$, $\dot{x}(0) = 0$. The reference curve shown on the graph is the solution for the critically damped case, which has a special property, as you're going to see.

4.1 Move the slider for b. Notice how the solutions for different b compare to the critically damped case. By experimenting, find the value of b such that the solution $x(t)$ gets small and *stays small* as rapidly as possible.

4.2 Give a mathematical explanation of the previous answer. Why does that value of b produce the fastest decay?

Lab 10: Tool Instructions

Damped Vibrations Tool

Setting Initial Conditions

Click the **[Start]** button to start a trajectory using preset initial conditions.
Clicking in the time series will set an initial value of *x* and start a trajectory.
Clicking in the plane while a trajectory is being drawn will start a new trajectory.

Parameter Slider

Use the slider to set the damping coefficient *b*.
Press the mouse down on the slider knob, and drag the mouse back and forth to change it, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the **[Pause]** button to stop a trajectory without canceling it.
Click the **[Continue]** button to resume the motion of a paused trajectory.

Critical Damping Tool

Parameter Slider

Use the slider to set the damping coefficient *b*. The slider controls the shape of the blue or red damping trajectory.
Press the mouse down for the parameter you want to change, and drag the mouse back and forth to change it, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled
[ ] **x**
[ ] **100x**
toggle the magnification level for viewing the damping behavior.