

9

Linear Oscillators: Free Response

Tools Used in Lab 9

Simple Harmonic Oscillator
Mass and Spring

Damped Vibrations: Energy

A mass bobbing on a spring, the quiet ticking of a grandfather clock, a child gently swinging under a tree—how can all these be modeled as unforced linear oscillators by means of homogeneous second-order linear differential equations with constant coefficients?

1. The Undamped Mass-Spring System

Open the **Simple Harmonic Oscillator** tool. The energy in a simple harmonic oscillator is completely determined by the initial position and initial velocity. There is no friction or external forcing. Set a variety of nontrivial initial conditions by clicking on the phase plane, then observe the motion in the phase plane and time series graphs.

The second-order differential equation that describes the mass-spring system is

$$m\ddot{x} + kx = 0, \quad (1)$$

where m , k , \ddot{x} , and x denote the mass, the spring constant, the acceleration, and the displacement of the mass from its equilibrium position, respectively. The sign of x is positive when the spring is stretched and negative when the spring is compressed.

- 1.1** In order to graph a second-order differential equation, it is necessary to rewrite it as a system of two first-order equations for velocity, $\dot{x} = v$, and acceleration, \dot{v} . Rewrite Equation (1) as

$$\dot{x} = v$$

$$\dot{v} =$$

- 1.2** As you observe the time series, note the relationship between the acceleration \ddot{x} and the displacement x . Where is acceleration \ddot{x} illustrated? How do you know this? Does this relationship hold for all initial conditions? Explain.

- 1.3** Describe how the displacement x and the velocity $v = \dot{x}$ are related. For what displacement is the velocity a maximum? a minimum?
- 1.4** Observe the phase plane. Note that the trajectories on the phase plane are always closed. What does this indicate about the motion?
- 1.5** Solve Equation (1) for $x(t)$, the displacement as a function of time. What is the frequency, ω , of the oscillation in radians per second?
- 1.6** Assume $m = 1$ and $k = 1$ for the tool. Calculate ω for these values.

1.7 Energy

- a. With no external forces, the only energy in the system is supplied by the initial displacement and the initial velocity. Look at the energy graph on the **Simple Harmonic Oscillator** tool.
- b. In general, the potential energy is the energy stored by the extension or compression of the spring. For a given displacement x , it is given by

$$E_{\text{potential}} = \frac{1}{2} kx^2.$$

What is the total energy of the system?

$$E_{\text{total}} = E_{\text{potential}} + E_{\text{kinetic}} = \underline{\hspace{2cm}}$$

- c. Will the total energy in this system dissipate as time increases? Explain.
- d. Calculate the total energy under the following initial conditions, for arbitrary k and m :

$$x(0) = 2$$

$$\dot{x}(0) = -1.$$

2. The Damped Mass Spring System

Open the **Mass and Spring** tool. Set some nontrivial initial conditions and observe the phase plane and the time series. The differential equation now includes a term for the frictional force:

$$F_{friction} = bv,$$

which is assumed to be proportional to the velocity of the mass. This would be the case for viscous damping (frequently illustrated with a “dashpot,” a plunger in a cylinder filled with viscous “goo”). The damping constant b is the constant of proportionality. The new equation is linear with constant coefficients:

$$m\ddot{x} + bv + kx = 0 \quad (2)$$

2.1 Rewrite Equation (4) as a system of first-order linear differential equations.

$$\dot{x} =$$

$$\dot{v} =$$

2.2 Describe the effect of damping on phase plane trajectories and the time series. Point out the differences in the graphs from those without damping.

2.3 From Equation (2) we can obtain the characteristic equation:

$$m\lambda^2 + b\lambda + k = 0$$

- Show that there are three possibilities: two real unequal negative values for λ , a repeated real negative root for λ , or a complex conjugate pair of roots. Show algebraically that the repeated root occurs when $b = \sqrt{4mk}$. This value of b is denoted b_c , the critical damping.

- b. Use the slider to try values of b that are above, at, and below b_c and observe the graphs. Label the three cases below with the appropriate terms: underdamped, overdamped, and critically damped. Make rough sketches of the time series for $x(t)$ for each of these cases, using the same initial conditions for comparison. State your initial conditions and show $x(0)$ appropriately on the sketch.

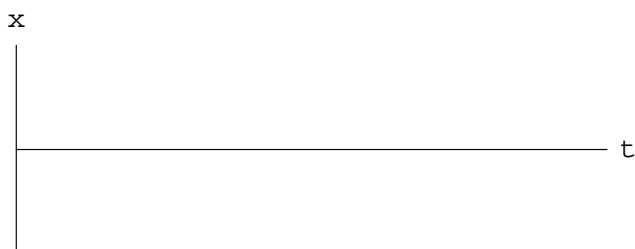
$$x(0) = \underline{\hspace{2cm}}$$

$$v(0) = \underline{\hspace{2cm}}$$

$$b < b_c \underline{\hspace{2cm}}$$



$$b = b_c \underline{\hspace{2cm}}$$



$$b > b_c \underline{\hspace{2cm}}$$



- 2.4 The Considerate Screen Door.** Suppose you wanted to design a damper on a screen door so that it would close as quickly as possible after being released. Due to the intense studying underway by your roommates, you also want it to close gently and smoothly without banging. What is the most desirable amount of damping?
- 2.5** Now that damping has been included in the model, we have dissipative heat losses due to friction. Open the **Damped Vibrations: Energy** tool and look at the energy graph. Vary the damping constant b to see how the energy loss due to friction is affected. The total initial energy is determined by the initial velocity, $\dot{x}_0 = v_0$, and the initial displacement, x_0 ,

$$E_{initial} = \frac{1}{2} k(x_0)^2 + \frac{1}{2} m(v_0)^2$$

How do you determine the energy loss due to friction?

Lab 9: Tool Instructions

Simple Harmonic Oscillator Tool

Setting Initial Conditions

Click the mouse on the $x\dot{x}$ phase plane to set the initial position and the initial velocity.

Click the mouse on the time series graph to set the initial position (initial defaults to zero). Clicking while a trajectory is being drawn will start a new trajectory.

Time Series Buttons

The buttons labeled

position

velocity

acceleration

toggle the time series graphs on and off.

Other Buttons

Click the **[Clear]** button to remove all trajectories from the graphs.

Click the **[Pause]** button to stop a trajectory without canceling it.

Click the **[Continue]** button to resume the motion of a paused trajectory.

Click the **[Energy Graph]** button to automatically display potential energy E_p and kinetic energy E_k whenever a trajectory is drawing the $x\dot{x}$ plane

Mass and Spring Tool

Setting Initial Conditions

Click the mouse on the $x\dot{x}$ phase plane to set the initial position and the initial velocity.

Click the mouse on the time series graph to set the initial position (initial defaults to zero). Clicking while a trajectory is being drawn will start a new trajectory.

Parameter Slider

Use the slider to set the damping constant b , m , and k . The dashpot will disappear when b is set to zero.

Press the mouse down for the parameter you want to change, and drag the mouse back and forth to change it, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled

position

velocity

acceleration

toggle the time series graphs on and off.

Other Buttons

Click the **[Clear]** button to remove all trajectories from the graphs.

Click the **[Pause]** button to stop a trajectory without canceling it.

Click the **[Continue]** button to resume the motion of a paused trajectory.

Click the **[Draw Field]** button to draw a slope field over the $x\dot{x}$ graphing plane.

Damped Vibrations: Energy Tool

Setting Initial Conditions

Click the **[Start]** button to start a trajectory using preset initial conditions.

Clicking in the time series will set an initial value of x and start a new trajectory.

Parameter Slider

Use the slider to set the damping coefficient b .

Press the mouse down for the parameter you want to change, and drag the mouse back and forth to change it, or click the mouse in the slider channel at the desired value for the parameter.

Time Series Buttons

The buttons labeled

position

velocity

acceleration

toggle the time series graphs on and off.

Other Buttons

Click the **[Pause]** button to stop a trajectory without canceling it.

Click the **[Continue]** button to resume the motion of a paused trajectory.