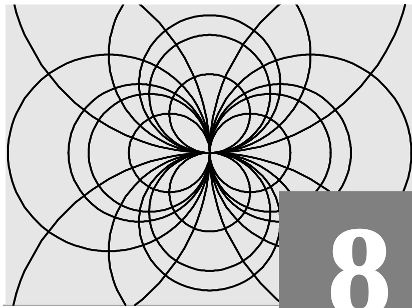


8

Orthogonal Trajectories

Tools Used in Lab 8
Orthogonal Trajectories

If two families of curves always intersect each other at right angles, then they are said to be orthogonal trajectories of each other. Given one family, can you predict the other?

1. Orthogonal Curves

Try a few equations in the **Orthogonal Trajectories** tool. Note that as a point on the plane is selected and clicked on, both the curve (yellow) and its orthogonal trajectory (blue) are given through the point.

If one family consists of the solution curves for

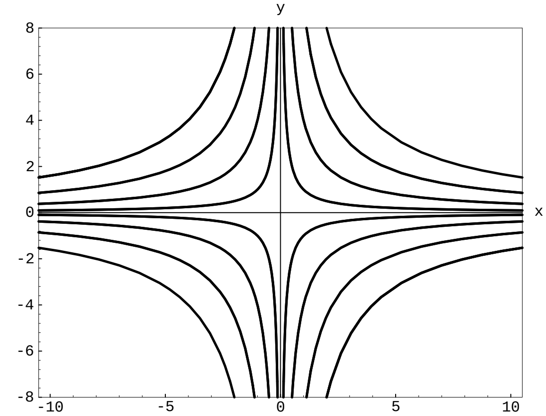
$$\frac{dy}{dx} = f(x,y),$$

then the family of orthogonal trajectories must be solution curves with orthogonal slopes. This means that they must satisfy a different differential equation:

$$\frac{dy}{dx} = \frac{-1}{f(x,y)}.$$

This is the general method. If we start with the equation for the family of curves, we must first differentiate to find the differential equation for the slopes, then find the differential equation for the orthogonal slopes.

1.1 Consider the family of curves $xy = C$. The curves are pictured below. On the same graph, sketch the orthogonal family. After you have drawn a few of these curves you will find it simple to predict orthogonal trajectories.



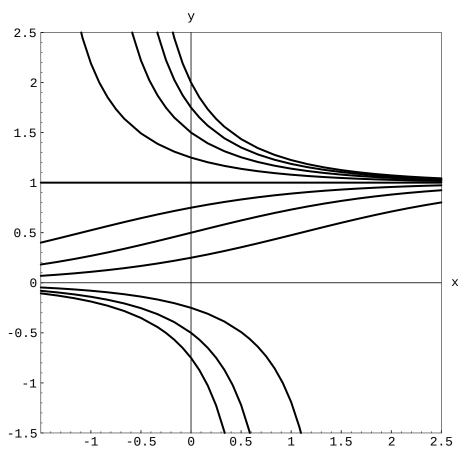
1.2 Find the equations for the orthogonal family of curves using analytical methods.

Hint: You should first use implicit differentiation on $xy = C$ to find $\frac{dy}{dx} = f(x,y)$ for the original curves. Then solve a new differential equation, $\frac{dy}{dx} = -\frac{1}{f(x,y)}$.

Now check your sketch with the **Orthogonal Trajectories** tool. Look at the list of differential equations to find the one with solution curves $xy = C$. Does your sketch agree? If not, you can fix it now.

1.3 As in Exercise **1.1**, visualize and sketch the graph of the orthogonal trajectories of the logistic equation: $\frac{dy}{dx} = y(1-y)$. Then check your prediction with the **Orthogonal Trajectories** tool.

What is the differential equation that the orthogonal trajectories must satisfy?

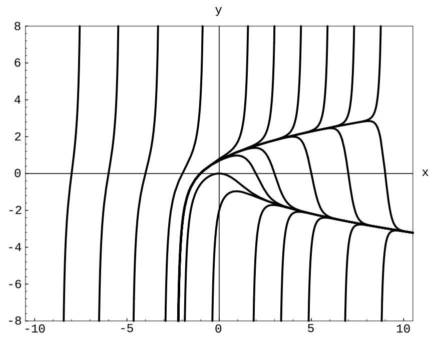


The **Orthogonal Trajectories** tool generates *two* families of mutually orthogonal curves, yellow and blue, respectively; each family can be found if you know the other.

In fact, most ODE solvers allow us to graph orthogonal trajectories in the following way. Start with the differential equation $\frac{dy}{dx} = \frac{h(x,y)}{g(x,y)}$. Define $\frac{dx}{dt} = g(x,y)$, $\frac{dy}{dt} = h(x,y)$ where x and y are viewed as functions of t. Now the trajectories in the *phase plane*, the (x,y)-plane, are the solution curves for the original differential equation $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{h(x,y)}{g(x,y)}$. We can redefine $\frac{dx}{dt} = -h(x,y)$ and $\frac{dy}{dt} = g(x,y)$ to obtain the slope $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x,y)}{h(x,y)}$ of the orthogonal trajectories which can now be graphed in the phase plane. It is possible to find two mutually orthogonal families of solution curves even though we may not be able to solve either of the corresponding differential equations algebraically. Several such cases appear in the **Orthogonal Trajectories** tool.

1.4 Match the pairs of families of orthogonal trajectories. Note that these are not screen pictures, but are on a larger scale, $-10 \le t \le 10$, $-8 \le t \le 8$, giving a larger view of solutions.

A.



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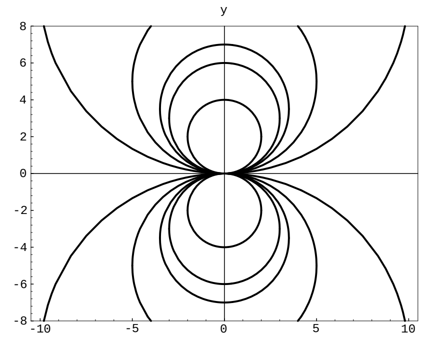

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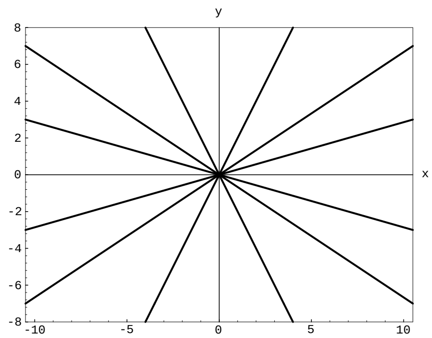
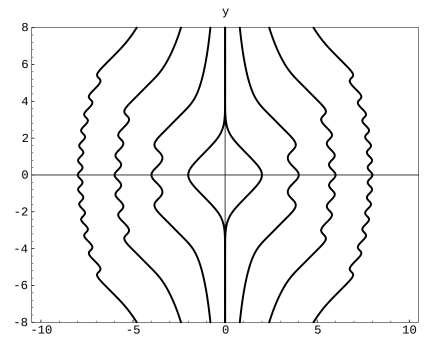
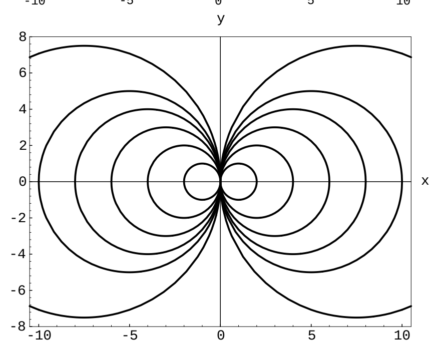


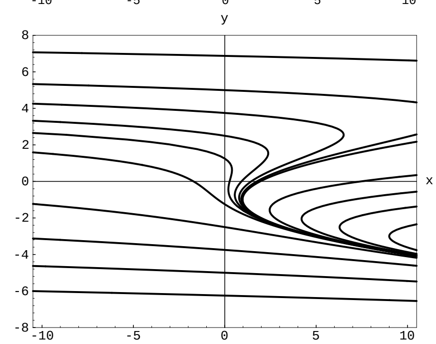
D.



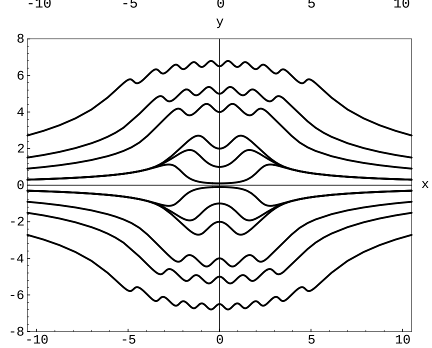
E.



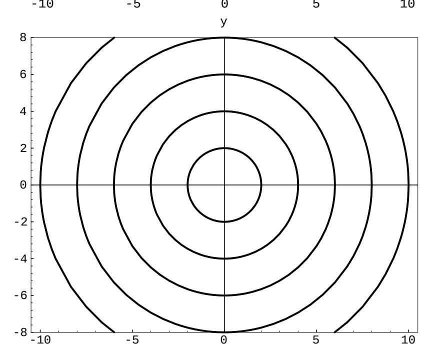
F.



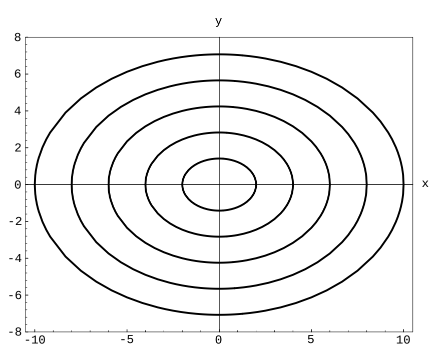
G.



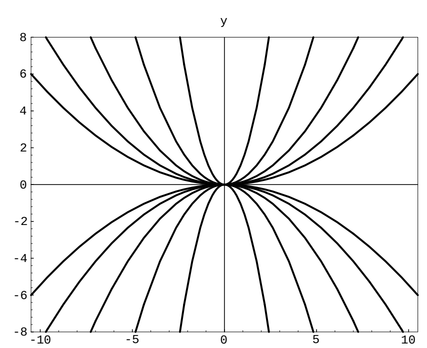
H.



I.



J.



1.5 Match the differential equations with the solution graphs in Exercise **1.4**.

____ a. $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2}$

____ b. $\dfrac{dy}{dx} = \dfrac{-x}{2y}$

____ c. $\dfrac{dy}{dx} = \dfrac{-x}{y}$

____ d. $\dfrac{dy}{dx} = \sin(xy)$

____ e. $\dfrac{dy}{dx} = y^2 - x$

____ f. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$

____ g. $\dfrac{dy}{dx} = \dfrac{2y}{x}$

____ h. $\dfrac{dy}{dx} = \dfrac{y}{x}$

____ i. $\dfrac{dy}{dx} = -\csc(xy)$

____ j. $\dfrac{dy}{dx} = \dfrac{1}{x - y^2}$

2. Some Practical Considerations

The electric field and equipotential lines around a charge are mutually orthogonal families of curves in the plane.

2.1 Which one of the families of curves in Exercise **1.4** represents the electric field lines around a point charge represented by a point in the plane? In three dimensions, these would be the field lines around a charged wire passing through the origin perpendicular to the plane.

2.2 Which family represents the equipotential curves? As above, these would actually be equipotential surfaces in three-dimensional space.

2.3 Can you identify the pair that represents the magnetic field lines about a dipole and the magnetic equipotentials?

Lab 8: Tool Instructions

Orthogonal Trajectories Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial conditions for a trajectory and its associated orthogonal trajectory.
Clicking while a trajectory is being drawn will start a new trajectory.

Equations

Click the button to the left of the equation to scroll the list of equations.
Click on an equation to select it.

Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from the graph.
Click the mouse on the **[Draw Field]** button to draw a slope field over the graphing plane.