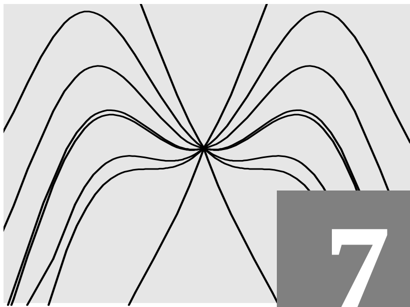


# 7 Existence and Uniqueness

**Tools Used in Lab 7**

- Targets
- Sure-Fire Target
- Uniqueness

*How do we know that a solution to a differential equation exists for a given initial value? Myriad solutions may exist, sometimes radiating star-like from a particular point, or there may be only one solution. In some cases there are none.*

## Target Shooting

Target shooting leads in a natural manner to concepts of existence and uniqueness of solutions to first order differential equations. There is, however, a limitation to the **Targets** tool that arises from the fact that a pixel on the screen has a discrete non-zero area. In the neighborhood of converging solutions a target may be hit by more than one trajectory. This spurious effect has been disguised somewhat by the tool designer.

Open the **Targets** tool. Select the differential equation $\dot{x} = x^2 - t$. Pick a point in the field to be a target and click on it with the mouse. Pick another point some distance away that you think may lie on the same solution curve. "Shoot" at the target by clicking on the second point. Keep selecting points from a sporting distance until you get a hit. The distance that is "sporting" varies with the behavior of solution curves in the neighborhood of the target. If the solution curves are diverging you may need to move quite close to the target to get a hit. If the solution curves are converging toward the target, you will be able to hit the target from much further out. To select a new target point, click the **Clear Target** button, then click the mouse on the desired spot. Try some other equations, especially the "tricky ones."

**Two Important Questions** (Not to answer now, but to keep in mind as you are doing this lab.)

1. **Can you always hit the target?** Is there always at least one solution through a given target point? That is, does a solution exist?

2. **Can you hit the target more than once?** Can there be more than one solution through a given target point? That is, is a solution unique?

# 1. Existence

The first important question is the question of **existence**.

Given $\dot{x} = f(t, x)$ and a point on the plane, is there a solution that passes through the point? The answer is *yes* if $f(t, x)$ is real-valued and *continuous in an open region containing the point*.

**1.1** Find at least two examples from the differential equations in the **Targets** tool where there are large regions in which no solutions exist. Describe the regions for each equation.

**1.2** Consider the differential equation $\dot{x} = x / \cos(t)$. For what values of $(t, x)$ can solutions *not* be guaranteed?

# 2. Uniqueness

The second important question is the question of **uniqueness.**

Given $\dot{x} = f(t, x)$ and a point on the plane, is a solution that passes through the point the only such solution? Now we add a new "smoothness" condition on $f(t, x)$ such as *continuous differentiability throughout an open region containing the point*. To experience non-uniqueness, try the **Sure-Fire Target** tool.

**2.1** Start with the differential equation $t\dot{x} = x$. From your observations, for what point or points in the plane do solutions exist? For what point or points are the solutions unique? (*Hint*: Sometimes you may be shooting "backwards in time"!)

**2.2** Solve the differential equation $t\dot{x} = x$ analytically.

Did you find a solution that passes through $(t, x) = (1, 0)$? What would it be? Note that $x \equiv 0$ is a valid solution. Modify your answer to Exercise **2.1** if necessary.

**2.3** Try $t\dot{x} = x + t^2 \cos(t)$ in the **Sure-Fire Target** tool. What kind of differential equation is this? Is it separable, exact, or linear? Solve this equation. Obtain all the solutions.

**2.4** For what points in the plane are there unique solutions? non-unique solutions? no solutions? Show, with your "family of solutions" in Exercise **2.3**, that any non-unique solutions you might have are justified.

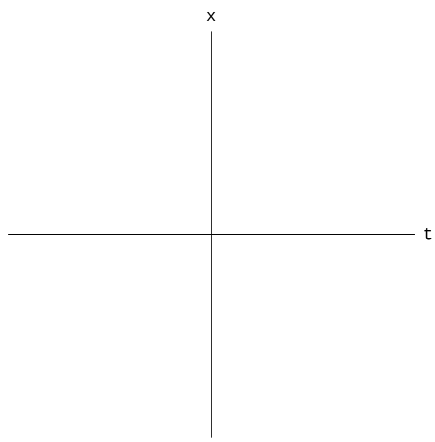
**2.5** Open up the **Uniqueness** tool. Consider the two examples $\dot{x} = x^{2/3}$ and $\dot{x} = x^{4/3}$. For what regions (for example, for $a < t < b$, $c < x < d$) can the existence of solutions be guaranteed for an initial point in the region? How must we further restrict the regions to be certain that the solutions are unique? For the analysis, check to see where, if at all, the functions $f(t, x)$ fail to be continuously differentiable (which requires that $\partial f / \partial x$ exists and is continuous).

The pictures for both differential equations are a bit tricky. To see the behavior of the solutions more clearly as they approach the $x$-axis, look at the vertical enlargement shown on the right. Note that one differential equation has distinct solutions everywhere, some of which are approaching the horizontal axis asymptotically, and the other has non-unique solutions on the horizontal axis. These results should agree with your analysis.

**2.6** The computer is somewhat selective about which solution it shows when there are many solutions through a point, say (1,0). If you pick a point on the horizontal axis, the solution follows the horizontal axis through (1,0); there are, however, many other solutions through this point. Note that solutions may stay unique in one region but be non-unique in larger regions. Remember, the solution is any smooth (differentiable) function that passes through (1,0) and satisfies the equation.

### What Happens as a Solution Crosses the t-Axis?

If you pick a point that is off the horizontal axis, the computer only shows the solution that crosses directly over the axis in the direction it was heading. However, there may be infinitely many ways for a solution to continue when it reaches the axis. For example, the solutions to $\dot{x} = x^{2/3}$ through $(1,0)$ may be obtained by hooking up pieces of solutions joined together (in proper fashion) by a horizontal piece along the $t$-axis containing $(1,0)$. Using colored pencils, sketch three solutions through $(1,0)$.




**2.7** Solve both of the following equations analytically. Compare the solutions with the graphical results. Is $x \equiv 0$ a valid solution for both? For each equation, are there other solutions that intersect the $t$-axis? What does this say about the uniqueness of $x \equiv 0$ ?

$\dot{x} = x^{2/3}$

$\dot{x} = x^{4/3}$

**2.8** Determine $\dfrac{\partial f}{\partial x}$ for both examples in Exercise **2.5**. Are your results consistent with the uniqueness test at the beginning of Section 2?

The questions of existence and uniqueness are not trivial mathematical questions. In fact, it is important to realize that most differential equations do *not* have explicit formulas for their solutions. Nevertheless, you have learned that with graphics, if solutions exist, you can see them, describe them, and make predictions about their behavior in the long term.

## 3. Additional Exercise

Note that even if a unique solution exists at every point, you might have a case where the solutions “blow up”—become infinite in a finite time. Look at the example $\dot{x} = 1 + x^2$ using the **Targets** tool**.** Solve it by separation of variables to obtain the analytical solution. Use this to explain what happens if you try to “shoot” at (0, 0) from ($a$, $b$) where $a > \frac{\pi}{2}$.

# Lab 7: Tool Instructions

## Targets Tool

### Setting Initial Conditions

The first click on the graphing plane will set a target. Once a target is set, click on another point to set the initial conditions $x(0)$ and $t(0)$ for a solution curve that you think will pass through the target point. Clicking while a trajectory is being drawn will start a new trajectory.

### Equations

Click the button to the left of the equation to scroll the list of equations.
Click an equation to select it.

### Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from a graph.
Click the mouse on the **[Clear Target]** tool to remove the active target point, then click on the plane to set a new target.
Click the mouse on the **[Draw Field]** button to draw a grid of vectors over the graphing plane.

## Sure-fire Target Tool

### Setting Initial Conditions

Click the mouse on the graphing plane to set the initial condition for a trajectory.
Clicking while a trajectory is being drawn will start a new trajectory.

### Equations

Click the button to the left of the equation to scroll the list of equations.
Click an equation to select it.

### Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from the graph.
Click the mouse on the **[Draw Field]** button to draw a slope field over the graphing plane.

## Uniqueness Tool

### Setting Initial Conditions

Click the mouse on the graphing plane to set the initial condition for a trajectory.
Clicking while a trajectory is being drawn will start a new trajectory.

### Equations

Click the button to the left of the equation to scroll the list of equations.
Click an equation to select it.

### Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from the graph.
Click the mouse on the **[Draw Field]** button to draw a slope field over the graphing plane.