

6

Isoclines and Fences

Tools Used in Lab 6

Isoclines
Isoclines as Fences

To the Instructor: Why bother with this topic? Traditional courses often omit isoclines, and only the newest texts might refer to fences and funnels. But both concepts are important in moving toward a useful geometric view of differential equations. Now that graphics have helped us to make the pictures part of a student's "world view" of differential equations, isoclines can be valuable in making a quick sketch when one is not at a computer or calculator. Fences and funnels make a simple and consistent way to approach many a proof, including many that actually attract students. See Section 2 for a bit of illustration.

Solutions in a slope field wander from isocline to isocline, leaping over fences, sometimes slipping out of antifunnels, sometimes sliding into funnels, never to escape! What are these notions?

1. Isoclines

An **isocline** is a curve on which the solutions to the differential equation have the same slope. For a given differential equation $\dot{x} = f(t, x)$, set the function $f(t, x)$ equal to a constant to define an isocline.

$$\dot{x} = f(t, x) = c$$

For any particular value of the constant c we usually obtain an equation for a curve, the isocline. However, there may be values of c for which no such curve exists.

- 1.1** Give an example of $\dot{x} = f(t, x)$ where $f(t, x) = c$ does *not* exist for some c . If $c = 0$, for example, can you find a function $f(t, x)$, where $f(t, x) = 0$ does not exist?

Why Are Isoclines Important?

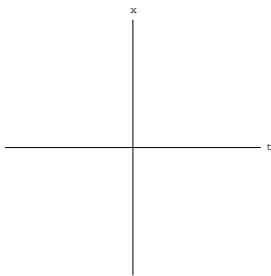
Before the widespread availability of computer solvers for differential equations, isoclines were used as an aid to graphing solutions. They are often the easiest way to sketch a solution by hand. Now they are more important as a check on the reliability of computer-generated solutions and as a way to gain insight into the long-term behavior of solutions (see Section 2). Remember that isoclines are *not* solutions (except in some unusual instances to be discussed later), so it is always a good idea to draw them in a different color than the solutions.

- 1.2** Using the **Isoclines** tool, choose the differential equation $\dot{x} = x^2 - t$, and, using the "slider" for c , find some isoclines. Include isoclines for $c = 0$ and for several positive and negative values of c . Note that the slope marks on each isocline denote the direction of the slope vectors of the solutions passing through the isoclines.

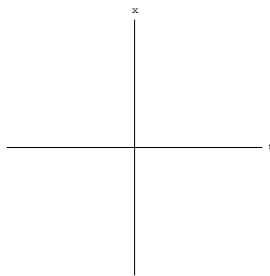
From your knowledge of the fact that solutions move tangent to the slope marks, try to predict the path of the solution through the point $(1,0)$. Now click on that point to find the solution through $(1,0)$. How did you do?

- 1.3** Try out the different equations in the **Isoclines** tool to see different isocline and solution behaviors. For each of the equations below, sketch a few of the solutions. Then write a short description of what the isoclines look like, and how the solutions behave.

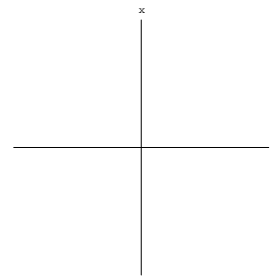
$$\dot{x} = x(1-x)$$



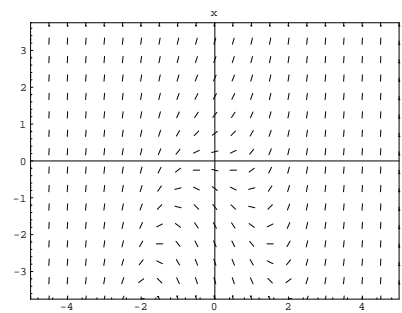
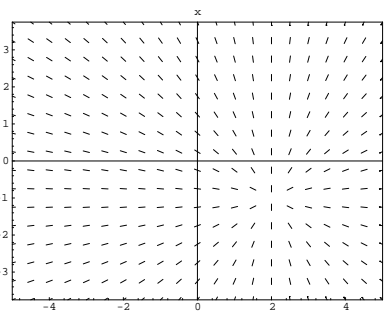
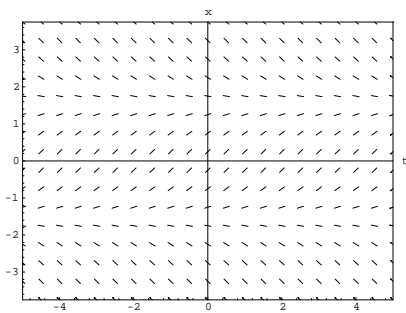
$$\dot{x} = t - 2x$$



$$\dot{x} = x^2 - t^2$$



A nullcline is an isocline for which $\dot{x} = 0$, that is, $f(t,x) = 0$. A nullcline sketch makes the sketching of solution curves much easier, and can be used to prove the behaviors of solutions. With a colored pen or pencil, locate the nullclines on the graphs for each of the equations in the **Isoclines** tool.



- 1.4** Find the nullcline formula for each differential equation below. Use colored pencils to sketch the nullcline and at least three isoclines on each slope field.

$$\dot{x} = \cos(x)$$

$$\dot{x} = \frac{x+1}{t-2}$$

$$\dot{x} = x + t^2$$

nullcline _____ nullcline _____ nullcline _____

- 1.5** Can you find an exception to the general rule that “an isocline is not a solution”? Look at the examples from Exercise 1.3. Confirm your finding algebraically.

2. Fences, Funnels, and Antifunnels

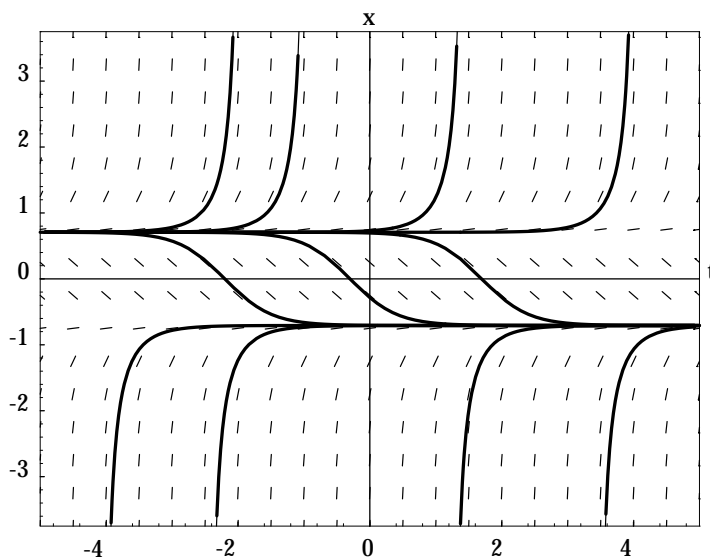
The ideas in the remainder of this lab—fences, funnels, and antifunnels—were developed by John Hubbard at Cornell University to make simple but rigorous proofs about the long-term behaviors of solutions to differential equations. They provide a vocabulary that allows easy descriptive statements and short proofs rather than lengthy arguments. Such qualitative techniques often provide surprisingly quantitative information, such as location of asymptotes to any desired accuracy. A detailed discussion can be found in [HW], Sections 1.3 through 1.6 and 4.7, with examples and applications beyond those discussed below.

Fences

Turn to the **Isoclines as Fences** tool, change the constant with the slider, and notice the different colors used to draw the isocline. The two colors denote **upper fences** and **lower fences**. A **fence** is some curve other than a solution that “channels” the solutions in the direction indicated by the slope field. Isoclines can be fences, but so can many other curves, as we discuss in the exercises.

Look at the screen for $\dot{x} = x^2 - t$. If we consider points of intersection between solutions and another curve (in this instance an isocline), the curve is an upper fence if the solution slopes are *less* than the curve slopes at the intersections, and is a lower fence if the solution slopes are *greater* than the curve slopes. Solutions move up from lower fences and down from upper fences, as they go from left to right.

- 2.1** Note that isoclines are not the only possible fences. Consider the slope field for $\dot{x} = 2x^2 - 1$ with some solutions sketched. With a pencil, draw a horizontal line and a couple of oblique lines across the picture. Then color parts of the lines according to whether they act as upper or lower fences.

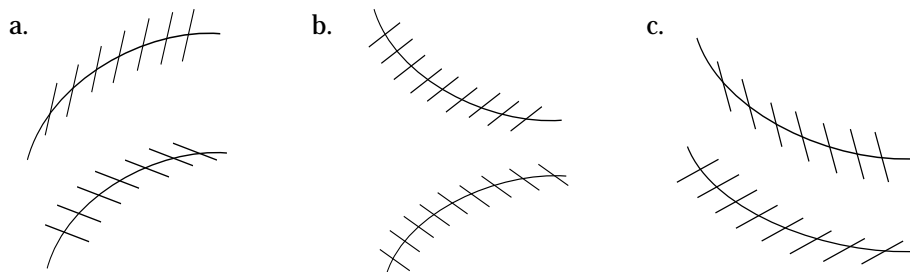


Funnels and Antifunnels

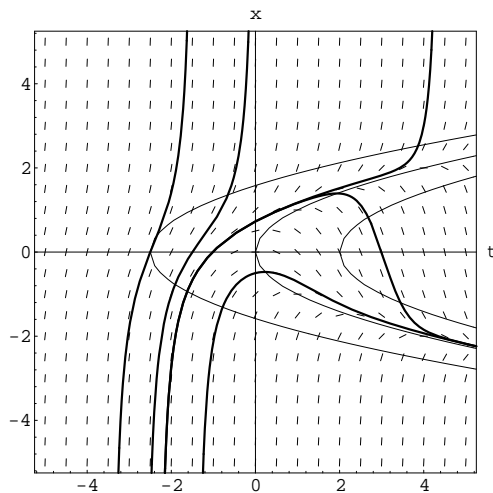
Visualize solutions converging toward a single solution as $t \rightarrow \infty$. We say the solutions are **funneled** toward the single solution and we describe the pattern of solution behaviors as a **funnel**. If an upper fence is always above a lower fence, the region inside is a funnel.

Alternatively, imagine solutions diverging or moving away from an exceptional solution. We describe this pattern of solution behaviors as an **antifunnel**. If an upper fence is always below a lower fence, the region inside is an antifunnel.

2.2 The fences for some funnels and antifunnels are sketched below. Which pairs are which? Label the sketches appropriately.



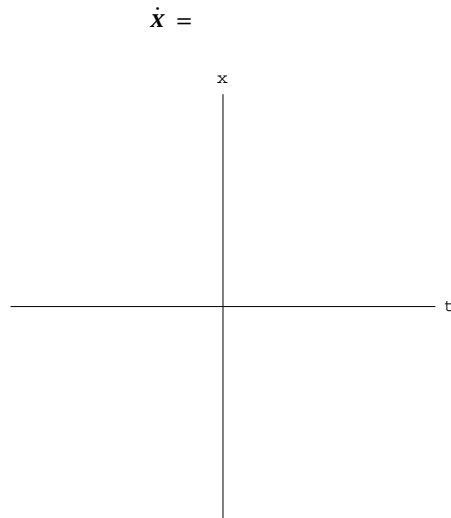
2.3 Some solution curves for $\dot{x} = x^2 - t$ are shown below, with isoclines shown as finer lines. With colored pencils, shade and label a narrowing funnel and a narrowing antifunnel.



A unique solution gets trapped in the narrowing antifunnel you have just highlighted. Add to your picture a sketch of the unique solution that separates solutions that go to positive infinity from those that converge in the funnel.

Defining the locations of the funnel and antifunnel allow us to describe the long term-behavior of solutions more precisely than “some of them blow up” or “others approach negative infinity”—we know exactly *how* they do this. We can say more clearly that “for $x > \sqrt{t}$, solutions blow up (apparently with vertical asymptotes), and for $x < \sqrt{t}$, solutions are funneled toward negative infinity at the rate of $-\sqrt{t}$.”

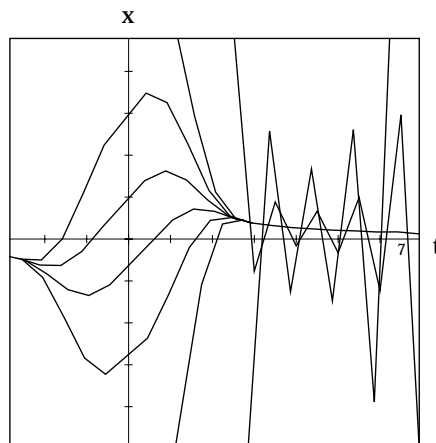
2.4 Make an analysis similar to Exercise 2.3 for at least one of the other equations in the **Isoclines as Fences** tool. Give a rough sketch and describe your results in the space below.



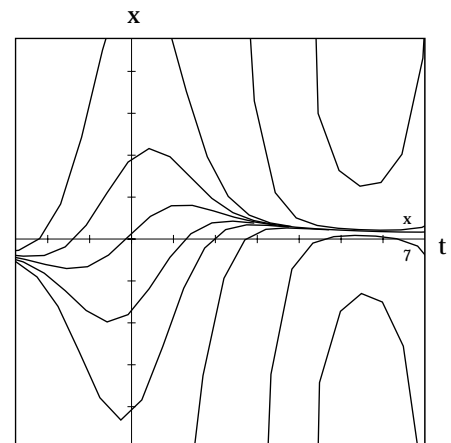
Why else, besides allowing precise descriptions, might fences and funnels be important or useful? Another example comes from the fact that pictures of approximate solutions may be misleadingly incorrect! For instance, the t -domain might be too large for the stepsize used in calculation, or the stepsize might be too large for the domain under study. Then you would want to recognize a mistake, and/or confirm a correct picture.

2.5 The equation $x' = 1 - tx$ is awkward at best to solve analytically, but it is easy to make pictures of the families of solutions. Each of the following pictures, however, is *wrong*, due to too big a stepsize (0.5) over too big a domain (with t going to 7).

Euler's Method

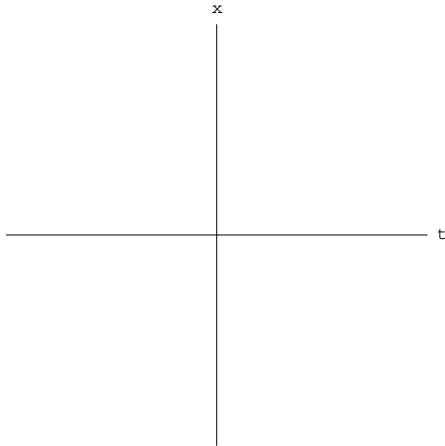


Runge-Kutta Method

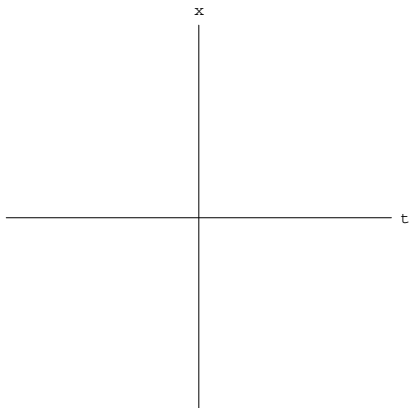


a. You should have an immediate clue that the picture from Euler's method on the left has at least some solutions that are not correct. Give a reason why.

- b. Label each of the Euler “solutions” with R for those that are approximately right and W for those that go wrong. Use the isoclines of slopes 0 and 1 to prove your answers, and indicate *where* the W solutions go wrong.



- c. From a first glance at the right-hand picture from the Runge-Kutta method, it is not obvious that there might be anything wrong, unless you have been thinking about the isoclines, or some other helpful aspect! If so, tell us, but don't worry if you have nothing to say here.
- d. Label each of the Runge-Kutta “solutions” with R for those that are approximately right and W for those that go wrong. Add to your graph the isoclines of slopes 0, 1, and -1 . (You'll want to think about other isoclines as well, but these provide the core of the argument.) Then make a fence and funnel argument to prove your answers, and indicate *where* the W solutions go wrong.
- e. Make a sketch of the correct solutions, with the isoclines of slopes 0, 1, and -1 . If you can, make observations or remarks due to additional information, e.g., from efforts to solve the equation.

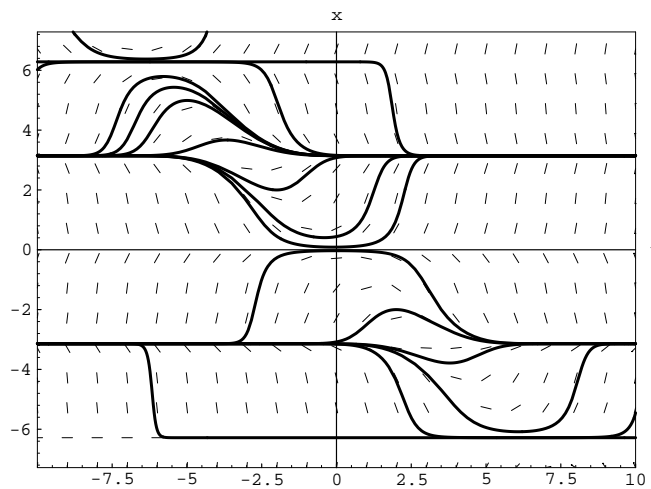


Moral: You cannot just believe a picture of a numerical approximation. In IDE we have tried to control the stepsizes and domains to avoid this kind of mistake. But when you are on your own using an open-ended graphing tool, with any old equation, you cannot be so sure. Isoclines, fences, and funnels are handy tools for checking out your graphs.

Check: Go back to the **Solutions** tool for Lab 2, Graphing Differential Equations. Note that for Euler with stepsize 0.5 you can get a recognizable error, for certain initial conditions!

3. Additional Exercise

- 3.1 For the differential equation with the slope field and solutions shown below, add fences that enable you to make an analysis and description of the solutions. These fences can be composed of lines or isoclines, or portions of either. (*Hint:* Find the oblique isocline of horizontal slope; then look to the right of that and try horizontal fences arbitrarily close to some interesting graphical features.)



Lab 6: Tool Instructions

Isoclines Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial condition for a trajectory.
Clicking while a trajectory is being drawn will stop the trajectory and draw a new one.

Parameter Sliders

Use the slider to set the constant slope c and reposition the isocline.
Press the mouse down on the slider knob and drag the mouse left and right, or click the mouse in the slider channel at the desired value for the parameter.

Equations

Click the button to the left of the equation to scroll the list of equations.
Click an equation to select it.

Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from a graph.
Click the mouse on the **[Draw Field]** button to draw a slope field over the graphing plane.

Isoclines as Fences Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial condition for a trajectory.
Clicking while a trajectory is being drawn will stop the trajectory and draw a new one.

Parameter Sliders

Use the slider to set the constant slope c and draw a new isocline.
Press the mouse down on the slider knob and drag the mouse left and right, or click the mouse in the slider channel at the desired value for the parameter.

Equations

Click the button to the left of the equation to scroll the list of equations.
Click an equation to select it.

Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from a graph.
Click the mouse on the **[Draw Field]** button to draw a slope field over the graphing plane.

