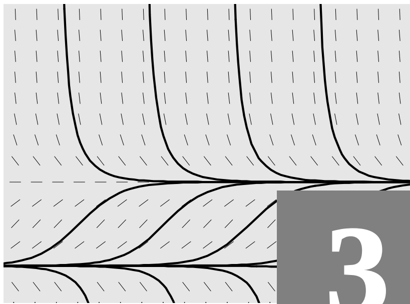


3

Single Species Population Models

Tools Used in Lab 3

- Growth and Decay
- Logistic Growth
- Logistic Phase Line
- Logistic with Harvest

Is there any way to predict how crowded our planet will be in the future? Population growth is often modeled by first-order differential equations. In this lab, we consider only autonomous differential equations—those without explicit time dependence.

1. The Exponential Model

The simplest model for growth and decay is the exponential model

$$\frac{dN}{dt} = rN\,, \qquad (1)$$

where r is a rate constant and N could be the size of a population, the density of bacteria in a nutrient medium, or a quantity of radium or carbon 14 in a particular state of radioactive decay. This model has solutions that are exponential functions.

1.1 Show the steps involved in finding the solution function $N = N_0e^{rt}$ for Equation (1), where $N_0 = N(0)$, the initial size of the population.

Open the **Growth and Decay** tool and try some negative values for r. The graph on the left is drawn numerically, using the differential equation, and the graph on the right is drawn using the solution function.

The time required for the amount of decaying material to be reduced by half is called the **half life** and is indicated in red along the solution curve. The time required for a quantity of carbon 14 to halve is about 5,568 years.

1.2 Write a formula for finding the half life. What is the value of r for carbon 14 decay?

For a population model, $N(t)$ is the number of individuals in the population at time t and r is a net growth rate, or birthrate minus death rate. Note that the solution to this differential equation is $N = N_0 e^{rt}$, so that for positive r the population grows without limit. For relatively small populations in a habitat with abundant resources, growth is exponential. The time required for a population to double is an important measure of growth. In the **Growth and Decay** tool, the doubling time is indicated in red alongside the solution function for positive values of r.

1.3 Write the formula for the doubling time of a population growing by the exponential model.

The Breakdown of the Exponential Model

In any real-world situation, totally unrestricted growth is clearly impossible. There are many limiting factors, not the least of which is the fact that the number of atoms in the solar system is finite.

2. The Logistic Equation

The logistic model is based on the exponential growth and decay model, but it includes an overcrowding term, or nonconstant growth rate, that reflects the limitations on growth due to the scarcity of resources and living space. The new term is proportional to the square of the population, so the equation becomes

$$\frac{dN}{dt} = rN - \frac{r}{K}N^2 \qquad \text{or} \qquad \dot{N} = r\left(1 - \frac{N}{K}\right)N \qquad r, K > 0\,. \qquad (2)$$

K is the steady state, or **carrying capacity**, for the population in a particular habitat. The term $\frac{r}{K}N^2$ is a mortality term in which N^2 represents competitive encounters between members of the population. Assuming positive values for N, we can also say that the impact of the growth rate r is diminished by $\left(1 - \frac{N}{K}\right)$ in proportion to the population size. Use the sliders in the **Logistic Growth** tool to observe the changes in the shapes of solutions for various values of the constants r, K, and N_0.

2.1 Describe the changes in population, if any, with increasing time for each of the following three relationships between the population density, N, and the carrying capacity, K:

$N < K$

$N = K$

$N > K$

2.2 Analyze the logistic equation as follows:

a. Find the equilibrium solutions for the logistic equation and justify this algebraically.

b. Explain why K is called the **carrying capacity** of the environment.

c. What method (or methods) can you use to solve the logistic equation analytically?

3. The Phase Line

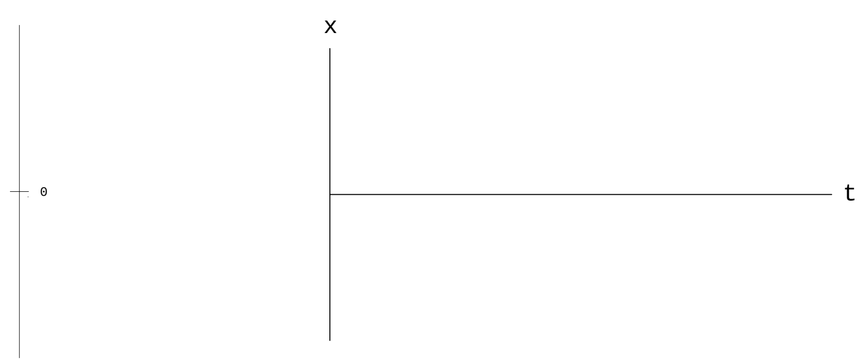
Open the **Logistic Phase Line** tool. The characteristic behavior of the logistic equation is displayed using $K = 1$ and $r = 1$. If x represents the population N, Equation (2) becomes

$$\dot{x} = x(1 - x)\,. \qquad (3)$$

Up to this point we have been plotting the tx graph and looking at the slope field defined by the slope $\dot{x}$ as a function of time. However, if we project the tx graph onto the x-axis (the vertical axis), we obtain the **phase line**, a succinct graphical summary of the behaviors of solutions**.** On the phase line, stable equilib rium points are shown as solid dots, unstable equilibrium points are shown as hollow dots, and the directions of flow are designated by arrows. Equilibrium points are also called **fixed points** or **steady states**, and are often designated symbolically with an asterisk, as in x^*. The phase line can also be viewed as a projection of the $x\dot{x}$ graph onto the horizontal axis. The $x\dot{x}$ graph is shown in the upper part of the screen. Clearly $\dot{x}$ as a function of x gives us a parabola that has zeros at $x = 0$ and $x = 1$, corresponding to the fixed points on the phase line.

The phase line is a way to view the **flow.** Think of a fluid flowing along the phase line with a velocity $\dot{x}$ that varies with the value of x according to $\dot{x} = x(1 - x)$, and imagine x as a particle swept along by the flow. As shown on the phase line, the flow is up where $\dot{x}$ is positive, down where $\dot{x}$ is negative, and there is no flow at the fixed points where $\dot{x} = 0$.

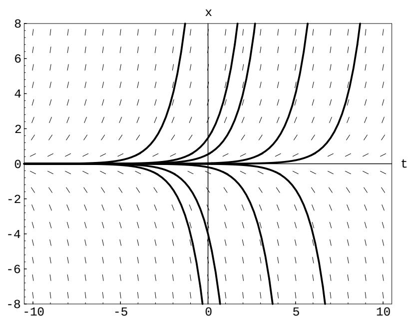
3.1 Sketch the graph of solutions to the logistic equation and give a general explanation in words of where stable and unstable equilibria are located. Then show how these are marked on the phase line.



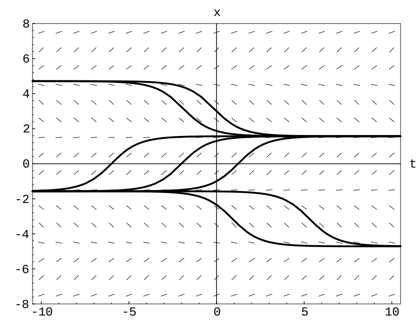
phase line

The **Logistic Phase Line** tool has shown one example of a phase line. In the following exercise, you will construct (by hand) phase lines for other equations.

3.2 For the following autonomous differential equations, use the given information (graph or equation) to locate on the phase line the equilibrium points (with a closed circle for a stable equilibrium and an open circle for an unstable equilibrium) and the appropriate arrows (according to whether x is increasing or decreasing in that region).



phase line a. $\dot{x} = x$



phase line b. $\dot{x} = \cos x$

The following, for which we have not given a picture, should be done simply by analyzing the signs that can appear in $\dot{x}$.

c. $\dot{x} = (x-1)(x-2)(x-3)$ phase line

$\dot{x} = \sin\left[\left(x - \frac{\pi}{2}\right)x\right]$ phase line

4. Logistic Growth with Harvesting

We can modify the logistic equation Equation (2) by including a harvesting term $h(t, N)$:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - h(t, N) \qquad (4)$$

For example, if our population were fish, $h(t,N)$ would represent a reduction in numbers due to fishing. Although seasonal variations in fishing might include an explicit dependence on t, we use the autonomous case in this lab. Another reasonable possibility is that the fishing rate might depend on the abundance of fish N. Using the simplified model Equation (3) with a **c**onstant harvesting rate h, we get

$$\dot{x} = x(1-x) - h\,. \qquad (5)$$

Using the **Logistics with Harvest** tool, observe the changes in the behaviors of trajectories when the value of h is varied.

4.1 Constant Harvesting Rate

a. Set h to 0.2. Look at the tx graph. Try several initial values for the population x by moving the cursor up and down on the phase line slider, or across the $x\dot{x}$ graph, to see how the initial population determines the future growth and survival of the population. When is there danger of extinction? Discuss the possibilities. Are there equilibrium populations? If so, what are they?

b. Set $h = 0.4$. Answer the preceding questions again. Are there any equilibrium populations? Can you find any initial population that does not result in extinction?

4.2 Critical Harvesting Rate

a. Experiment with h, starting with $h = 0.4$. Decrease h and look at the graph as well as the phase line. Note that the changes in h affect the number of fixed points and their stability.

   Choose h values that give qualitative differences in behavior, and draw the fixed points on the sketches of the phase line. Indicate whether they are stable, unstable, or semi-stable (denoted by a half-filled circle, a semi-stable point attracts on one side and repels on the other). If there are no fixed points, say so! Use arrows to show the flow directions along the phase line, toward or away from any fixed points.

   $h =$ $h =$ $h =$

   phase line phase line phase line

b. Describe what the number and value of fixed points signify about fish populations, over-harvesting, and survival vs. extinction in terms of our model.

   i. two fixed points

   ii. one fixed point

   iii. no fixed points

c. You have just found the critical **harvesting rate**, h_c, the harvesting rate for which there is only one fixed point. The change in the character of the system's behavior across this parameter value is called a **bifurcation.** What is your experimental result for h_c in the simplified model described by Equation (5)?

d. Now consider the general case, Equation (4). To find equilibrium levels (or fixed points) N_1* and N_2* you must set $\frac{dN}{dt} = 0$ and solve the resulting quadratic equation in N, using a constant h. Show your work.

e. Look at the discriminant using the quadratic formula. Show that N_1 and N_2 are positive if $h < rK / 4$ and that there is **exactly one** equilibrium level when $h = rK / 4$. Consequently this value of h must be the critical harvesting rate, h_c. Compare this value to your experimental value for h_c (when $r = 1$ and $K = 1$). What is the inescapable result when $h > h_c$?

4.3 Critique of the Model

a. What is wrong with the model? When does it not make sense?

b. Is the following model from Strogatz [SS, p. 90] more reasonable? Justify your answer.

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - h\frac{N}{A + N} \quad \text{where } h > 0 \text{ and } A > 0.$$

c. What happens when N gets large? When N gets small?

Lab 3: Tool Instructions

Growth and Decay Tool

Setting Initial Conditions

Click the mouse on the left graphing plane to set the initial conditions for a trajectory and the constant for the solution function.
Clicking while a trajectory is being drawn will stop the trajectory.
When you pass the mouse over the right plane, the functional relationship between the two variables is shown.

Parameter Slider

Use the slider to set the rate constant r.
Press the mouse down on the slider knob for the parameter and drag the mouse back and forth, or click the mouse on the slider channel at the desired value for the parameter.

Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from the graphing planes.
Click the mouse on the **[Drawing Field]** button to draw a slope field over the left graphing plane.

Logistic Growth Tool

Parameter Sliders

Use the sliders to change the values for the constants r, K, and N_0.
Press the mouse down on the slider knob for the parameter you want to change, and drag the mouse up and down, or click the mouse in the slider channel at the desired value for the parameter.

Logistic Phase Line Tool

Setting Initial Conditions

Click the mouse on the tx plane, or the $x\dot{x}$ plane or the phase line to set the initial condition for a trajectory.
Clicking while a trajectory is being drawn will stop the trajectory.

Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from the graphing planes.

Logistic with Harvest Tool

Setting Initial Conditions

Click the mouse on the tx plane, or the $x\dot{x}$ plane or the phase line to set the initial condition for a trajectory.
Clicking while a trajectory is being drawn will stop the trajectory.

Parameter Slider

Use the slider to set the harvest constant h.
Press the mouse down on the slider knob for the parameter and drag the mouse back and forth, or click the mouse in the slider channel at the desired value for the parameter.

Buttons

Click the mouse on the **[Clear]** button to remove all trajectories from the graphing planes.