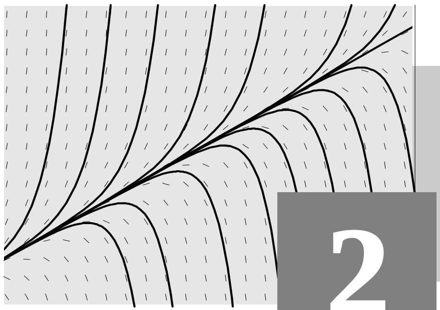


2 Graphing Differential Equations

Tools Used in Lab 2

Slope Fields
Solutions

A first-order differential equation is an equation for which the highest-order derivative is first-order. The goal is to find the function, knowing its derivative. How can we do this with graphs?

Introduction

A first-order differential equation has a whole family of solution curves, like a pot of spaghetti with waves of nonintersecting strands. With the tools for this lab, you can investigate such families, and also examine individual members of a family of solutions.

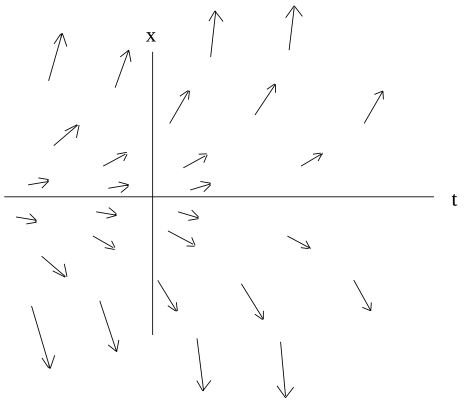
We purposely use t as the independent variable, because it is often helpful (and usually appropriate in context) to think of solutions evolving in time.

1. The Slope Field (or Direction Field)

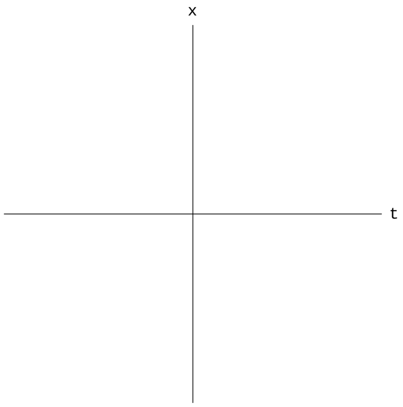
A first-order differential equation, $\frac{dx}{dt} = f(t, x)$, gives us, for any point (t,x) on the tx-plane, the **slope** of a solution curve. Just as a flag reveals the particular direction of the air current at a flagpole, $\frac{dx}{dt}$ reveals the direction (slope) of a solution curve $x = g(t)$ at any point (t,x) we choose. An aerial view of flags on a grid of flagpoles in a field would show the overall pattern of different wind currents in the field. Similarly, vectors with the slope $\frac{\Delta x}{\Delta t}$, on a grid of points on the tx-plane, indicate the flow of solution curves. Although $\frac{dx}{dt} = f(t, x)$ does not tell us directly the solutions $x = f(t)$, it does give the *slope* of the solution curve at any point. We know from differential calculus that the difference quotient $\frac{dx}{dt}$ gives a tangent line approximation of the curve $x = g(t)$ at every point on the curve. Our aim is to use $\frac{\Delta x}{\Delta t}$ as an approximation to $\frac{dx}{dt}$ to see the flow pattern of the solution curves.

1.1 With the **Slope Fields** tool you can see at any point (t,x) a vector with slope $\frac{dx}{dt}$. The slope calculation is shown in a separate window. By clicking the mouse, you can plot a segment of the slope vector proportional to the size of the time step. Note how the slope at each point is calculated from the formula for $\dot{x}$. All vectors indicate the direction of advancing time, t.

Choose one of the given equations to study, and make a picture of the slope field by setting down enough vectors to give an idea of what will happen wherever you might choose a starting point (t,x). Sketch the result here:



Our Example: $\dot{x} = x$



Your Example: $\dot{x} =$

1.2 Now the object is to find out what happens to a point set in this slope field. What path will be followed as a point is carried along by the flow, especially in the long run? The focus of studying differential equations is to be able to predict what will happen, given sufficient starting information.

To *see* a solution—the graph of the function that follows the flow—choose an initial point (t,x) on the slope field and click the mouse there to plot an arrow. At the head of that arrow, start another one, and so on, to build a crude approximation to a solution, $x = g(t)$. Your choice of an **initial condition** (t_0,x_0) picks out a unique member of the family of solutions.

You can click on the **[Draw Field]** button to get a whole grid of direction lines. Notice how your approximate solution follows all the little lines in the slope field. Fill in a drawing of an approximate solution on the slope field you have drawn in Exercise **1.1**.

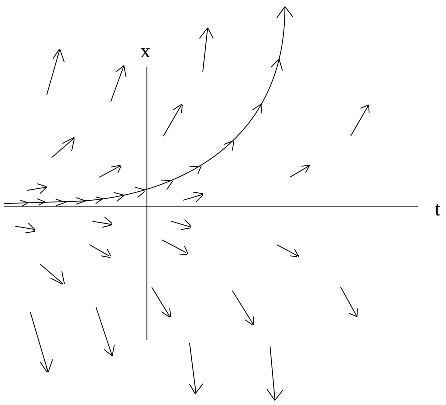
1.3 Try changing the choice of the time step, Δt. What happens to the size of the vector when you change from $\Delta t = 0.5$ to 1.0?

to 0.1?

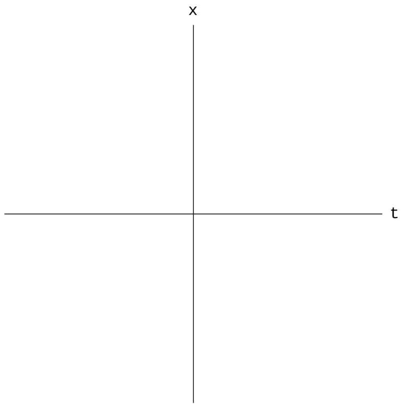
to 0.01?

Why do you end up with nothing but an arrowhead in the last case?

1.4 Now you are ready to let the computer do more of the work. If you click on the **[Solutions]** button, under **Drawing Mode**, every click on the slope field draws from that point an approximate solution computed with a small time step, Δt. By choosing more initial conditions, you can cover the *tx*-plane with as many solutions as you like to show what behaviors can be expected of the solutions for your equation, in the region defined by the graph. Thus you create a picture of the whole *family* of solutions to this particular differential equation. Sketch here a sampling of the solutions to your example.



Our Example: $\dot{x} = x$

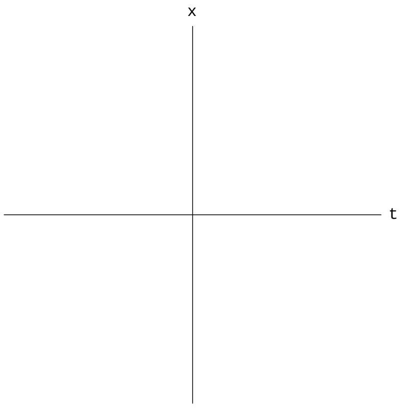


Your Example: $\dot{x} =$

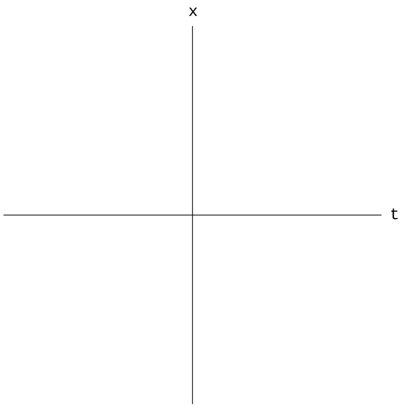
1.5 Write a verbal description of the behaviors of the solutions for your example, answering the following questions:

- How do these graphical solutions differ? by a constant? otherwise?
- Are there horizontal translates? vertical translates?
- Do they merge? do they diverge? where, and how?
- Do any solutions appear to have vertical asymptotes? horizontal asymptotes?
- Do any solutions appear to be **equilibria** (to have a constant value of x)? Do they connect to any of your answers above?

1.6 Look at a few more of the equations in the **Slope Fields** tool with the preceding questions in mind. Show one as an example, with your comments. Then, if you have not looked at $\dot{x} = \cos x$, try it now. Predict where the equilibrium solutions are.



$\dot{x} =$



$\dot{x} = \cos x$

2. Introduction to an Open-Ended Differential Equations Graphing Tool

An open-ended tool, which could be

- a graphing calculator
- a dedicated interactive differential equations program, such as *MacMath, Differential Systems, MDEP, Phaser**
- a computer algebra system such as *Derive, Maple, Mathematica,* or *Matlab**

lets you enter a function of your own choice for the differential equation, and change the bounds on the window. It also allows you to change the numerical approximation method and/or the stepsize to see the results.

Try whatever open-ended tool is at hand on one of the differential equations already studied (or on something different if you choose), to learn what it can do. Then you will henceforth have at your disposal a means to explore examples beyond those we have selected for the labs.

Make a print (or sketch) for your example, to show the effects of two different stepsizes, with Euler's method, on the same set of initial conditions. Annotate your printout with a little discussion of the results and how you think they could be explained.

3. Comparison with Algebraic Solutions

You have probably noticed that we have used no algebra, no calculus, and no obvious numerics up to this point, but rather have concentrated on qualitative pictures and ideas.

The actual way in which the solutions are drawn on the tx-plane is by numerical calculation of very short vectors using the slopes and small time steps. This is discussed in some detail in Lab 5, Numerical Methods.

3.1 We now turn our attention to those cases (a minority, except in differential equations textbooks) in which it is possible to find an algebraic formula for solutions by such methods as separation of variables or integrating factors for linear equations. This is called an **analytical** solution. As an example, we consider $x' = x - t$, for which the solution is $x = t + 1 + Ce^t$. Confirm that this is in fact the solution, either by analytical methods, or simply by differentiating to show that it indeed satisfies the differential equation:

3.2 Using the **Solutions** tool, choose an equation and start a solution; the computer then automatically draws both an approximate solution and the analytical solution (in different colors) through the same initial condition, after calculating the proper value of the constant C. Usually these curves look similar, but not exactly the same. How can you show that they differ? The kinds of differences you see depend on the example you have chosen.

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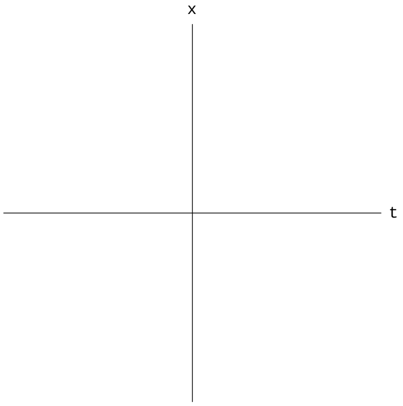
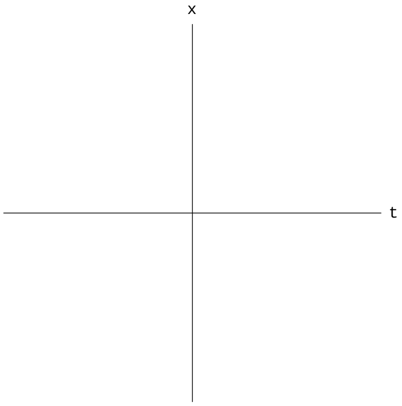
*For bibliographic details on any of the tools listed, see the list of resources in the front matter of this book.

First for $\dot{x} = x - t$, then for $\dot{x} = x \cos t$, set stepsize to 0.5 and method to Euler. Make a two-colored sketch to show what differences appear. For instance, for a given initial condition, are the intercepts at different coordinates on the approximate and analytical solutions? Or do they hit the edges of the graphs at different values? Do their vertical distances grow consistently?



Equation: $\dot{x} = x - t$



Equation: $\dot{x} = x \cos t$

Note: You may see numerical approximations become downright *wrong* when the timestep is too big for the time axis. A large time step occasionally causes sharp zigzags that look, and are, obviously, not correct. Such “jaggies” are illustrated in Lab 6, Isoclines and Fences, Exercise **2.5**.

3.3 For either $\dot{x} = x + t$ or $\dot{x} = x \cos t$, make a table listing several initial conditions and whatever measure seems appropriate to illuminate the difference(s),for example, t or x intercepts for approximate and algebraic solutions, or final values for t or x, or maximum vertical distance between approximate and algebraic solutions. Add whatever else you might wish to compare, such as values of the constant C, or the long term behavior of solutions.

your equation: initial conditions		comparison: intercepts or ________________			
t_0	x_0	approximate	analytical		

3.4 Write a paragraph about what you can observe from this experiment. Where is the approximate solution close to the analytical solution? Where is it not close?

As you study differential equations, you should become able to predict when the approximate numerical solution, computed in time steps, will not be a good fit to the analytical solution, computed algebraically.

3.5 With the solutions tool on these equations, try smaller stepsizes, and try Runge-Kutta approximations. Discuss the sort of improvements you seem to find for each.

Smaller stepsizes:

Runge-Kutta:

You will further explore these improvements in Lab 5, Numerical Methods.

3.6 Following is the list of the equations in the **Slope Fields** tool. For each of them, find the analytical (algebraic) solution where possible; otherwise, mark them as “not possible.”

1) $\dot{x} = x$

2) $\dot{x} = \cos t$

3) $\dot{x} = \cos x$

Does your solution to 3) include the equilibrium solutions found in Exercise **1.6**? These must often be found separately.

4) $\dot{x} = x \cos t$

5) $\dot{x} = \cos tx$

6) $\dot{x} = x + t$

7) $\dot{x} = x - t$

8) $\dot{x} = x^2 - t$

9) $\dot{x} = x^2 - t^2$

10) $\dot{x} = x^2 + t^2$

11) $\dot{x} = \cos(x^2 + t^2)$

12) $\dot{x} = -tx$

For each of these functions that has an analytical solution, the graphs of the numerical and analytical solutions can be compared with the **Solutions** tool. For those that do not, we have to rely on the graphical pictures of the solutions, but studying the other equations may start to give you some intuition about the accuracy of the numerical solutions in those cases as well.

3.7 For the equations in Exercises **3.6** where you know an analytical solution, explain how the formula confirms your visual expectation or forces you to revise some of the answers you made in Section 1. Comparison of analytical and numerical solutions can be used to advance your understanding of mathematics—so check carefully how the formulas compare with your descriptions, and acknowledge where refinements can now be made. Use a separate page if necessary.

4. Follow-Up Exercises for Classwork

Examine the twelve slope fields, with graphical solutions, as studied in the **Slope Fields** tool. Pictures are provided on the next two pages.

4.1 Solution curves that have identical shapes but different vertical and / or horizontal positions are called **translates**. Which of these slope fields have some solutions that are

horizontal translates?

vertical translates?

4.2 Which of these slope fields have

solutions with vertical asymptotes?

solutions with horizontal asymptotes?

solutions with oblique asymptotes?

merging solutions?

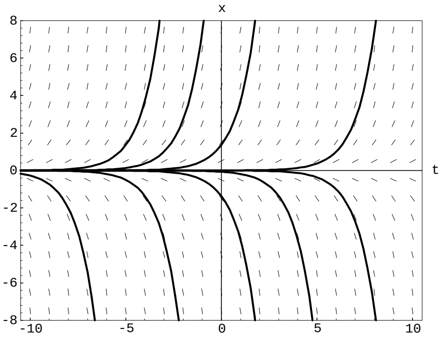
diverging solutions?

4.3 Can you make some conjecture(s) from the pictures about how to predict any of the above?

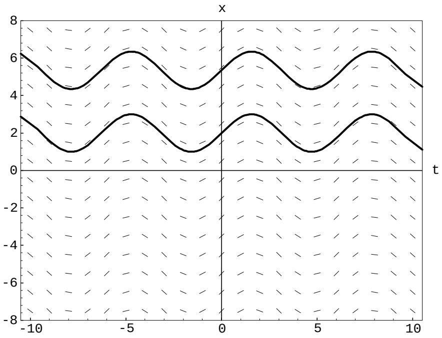
5. Slope Field Graphs

Pictures are provided of the twelve slope fields for equations in the **Slope Fields** tool. Here we use a larger "window" ($-10 \le t \le 10, -8 \le x \le 8$) than in the tool ($-2 \le t \le 2$, $-2 \le x \le 2$), in order to show more of the overall pattern.

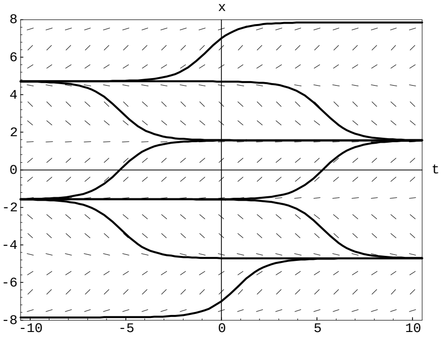
1) $\dot{x} = x$



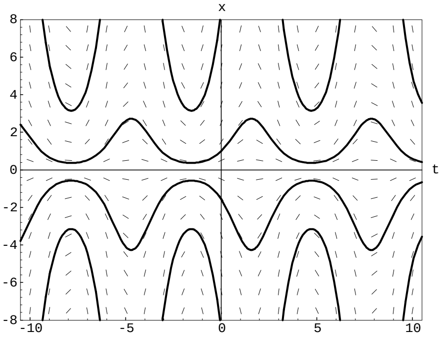
2) $\dot{x} = \cos t$



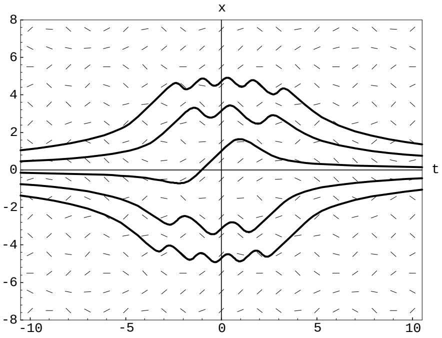
3) $\dot{x} = \cos x$



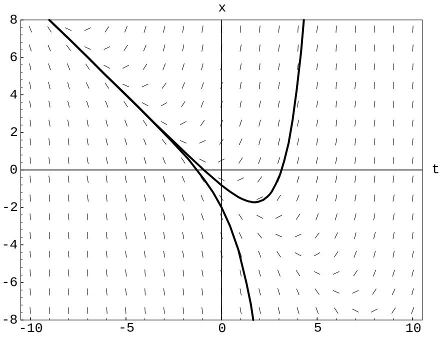
4) $\dot{x} = x \cos t$



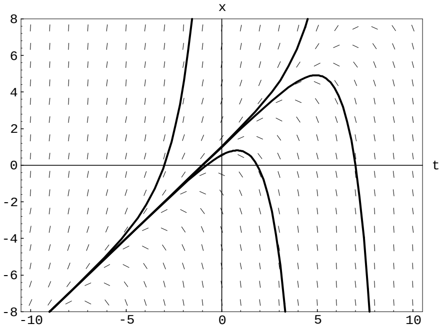
5) $\dot{x} = \cos tx$



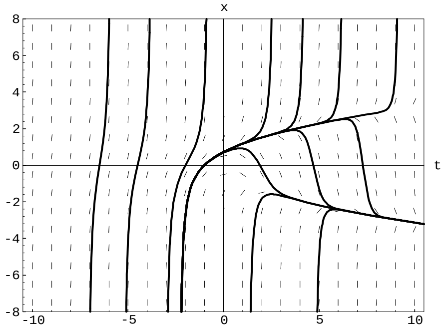
6) $\dot{x} = x + t$



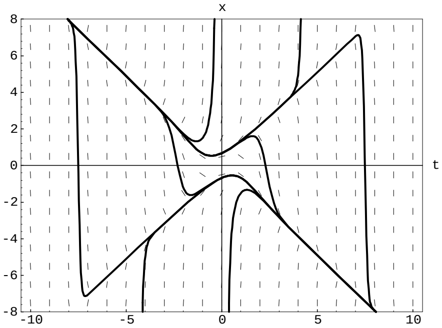
7) $\dot{x} = x - t$



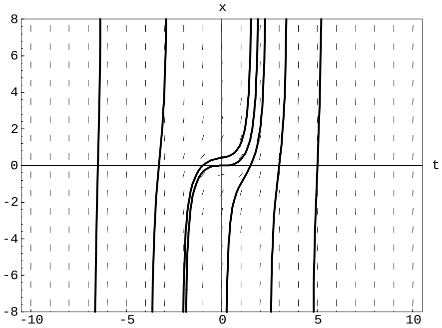
8) $\dot{x} = x^2 - t$



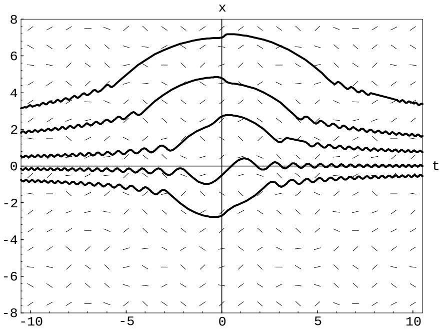
9) $\dot{x} = x^2 - t^2$



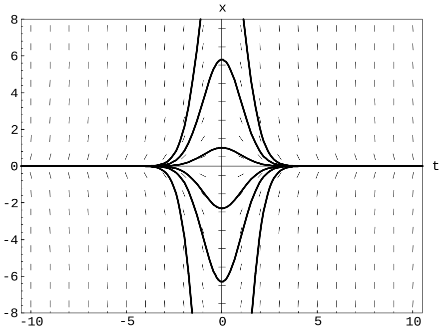
10) $\dot{x} = x^2 + t^2$



11) $\dot{x} = \cos(x^2 + t^2)$



12) $\dot{x} = -tx$



6. Additional Exercises

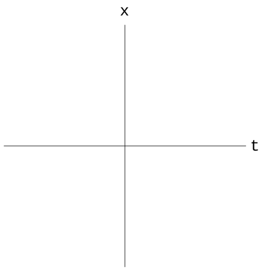
6.1 Use an open-ended differential equations graphing tool to make an analysis similar to that in Sections 1, 3, and 4 for the following equations:

a. $\dot{x} = -\frac{1}{x^2}$

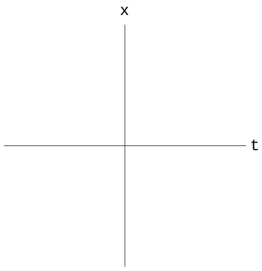
b. $\dot{x} = x^3$

6.2 Based on the similar equations in the **Slope Fields** tool, predict the behavior of the solutions for the following equations. Then use your open-ended differential equations graphing tool to make pictures of the solutions to at least one of the following equations. On a separate page, make sketches of the results, and add annotations that explain where your predictions were fulfilled, and where they were not. In the latter case, note what it was that made the picture differ from your expectations.

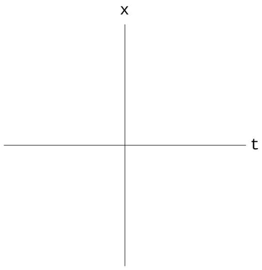
c. $\dot{x} = x / 2$



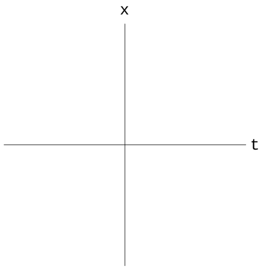
d. $\dot{x} = tx$



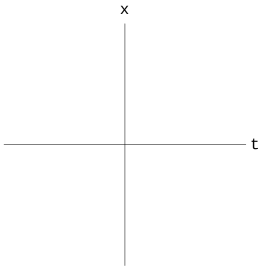
e. $\dot{x} = \cos 2t$



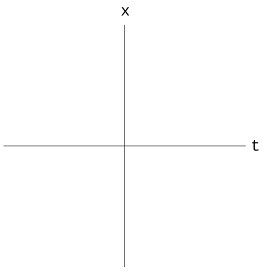
f. $\dot{x} = 2\cos t$



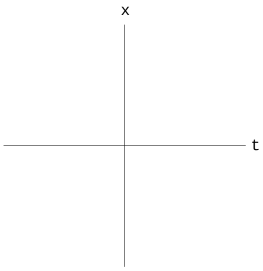
g. $\dot{x} = \sin x$



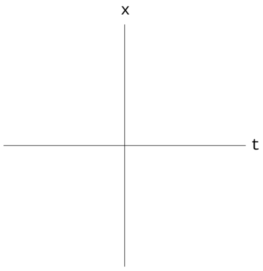
h. $\dot{x} = -\cos x$



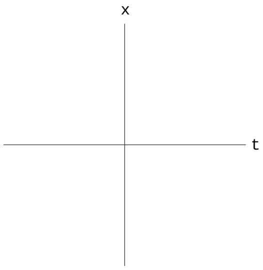
i. $\dot{x} = \sin tx$



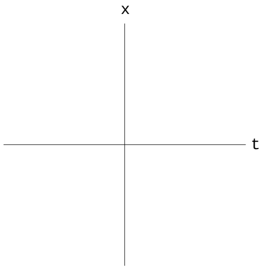
j. $\dot{x} = x^2 + t$



k. $\dot{x} = 2x + t$



l. $\dot{x} = x + 2t$



Lab 2: Tool Instructions

Slope Fields Tool

Setting Initial Conditions

Click the mouse on the graphing plane to set the initial conditions for a trajectory or a point for a vector.

Clicking while a trajectory is being drawn will stop the trajectory.

Equations

Click the arrow button to the left of the equation to pop up the list of equations.

Click an equation to select it.

Drawing Mode Buttons

Click the mouse on the **[Vectors]** button to set vectors when you click on the plane.

Click the mouse on the **[Solutions]** button to display a solution curve when you click on the plane.

Time Step Buttons

Click the mouse on a button in the Δt list to set the time step for vectors and trajectories.

Other Buttons

Click the mouse on the **[Draw Field]** button to draw a slope field over the graphing plane.

Click the mouse on the **[Clear]** button to remove all vectors and trajectories from the graphing plane.

Solutions Tool

Setting Initial Conditions

Click the mouse on the *tx* graphing plane to set the initial conditions for trajectory and define the constant for the solution function.

Clicking while a trajectory is being drawn will stop the trajectory.

When you pass the mouse over the *tx* plane the functional relationship between the variables is shown.

Equations

Click the arrow button to the left of the equation to pop up the list of equations.

Click an equation to select it.

Time Step Buttons

Click the mouse on a button in the Δt list to set the time step for the trajectories.

Other Buttons

Click the mouse on the **[Draw Field]** button to draw a slope field over the graphing plane.

Click the mouse on the **[Clear]** button to remove all slopes and trajectories from the graphing plane.

Click the mouse on the **[Euler]** button to draw a solution by Euler’s method.

Click the mouse on the **[Runge Kutta 4]** button to draw a solution using the Runge Kutta technique.