Harvesting Cod

Note: This exercise goes with Section 6.2 of Calculus with Applications.

A recent article described the population $f(S)$ of cod in the North Sea next year as a function of this year's population S (measured in thousands of tons) by various mathematical models:*

$$\text{Shepherd}: \quad f(S) = \frac{aS}{1+(S/b)^c};$$
$$\text{Ricker}: \quad f(S) = aSe^{-bS};$$
$$\text{Beverton}-\text{Holt}: \quad f(S) = \frac{aS}{1+(S/b)},$$

where a, b, and c are constants. The Shepherd model is the same as the model described in Exercise 54 of Section 2.4, $f(x) = \dfrac{\lambda x}{1+(ax)^b}$, if we replace S with x, a with λ, b with $1/a$, and c with b.

a. Find a replacement of variables in the Ricker model above that will make it the same as the Ricker model described in Exercise 38 of Section 6.2, $f(S) = Se^{r(1-S/P)}$.

b. Find $f'(S)$ for all three models.

c. Find $f'(0)$ for all three models. From your answer, describe in words the geometric meaning of the constant a.

d. The values of a, b, and c reported in the article for the Shepherd model are 3.026, 248.72, and 3.24, respectively. Find the value of this year's population that maximizes next year's population using the Shepherd model.

e. The values of a and b reported in the article for the Ricker model are 4.151 and .0039, respectively. Find the value of this year's population that maximizes next year's population using the Ricker model.

f. Explain why, for the Beverton-Holt model, there is no value of this year's population that maximizes next year's population.

g. In Exercise 38 of Section 6.2, we defined the harvest as $H(S) = f(S) - S$, and showed that the maximum sustainable harvest occurred when $f'(S) = 1$. Find the value of S for the Shepherd model that gives the maximum

* R. M. Cook, A. Sinclair, and G. Stefánsson, "Potential Collapse of North Sea Cod Stocks," *Nature,* Vol. 385, Feb. 6, 1997, pp. 521-522.

sustainable harvest, using the values of a, b, and c given in part d. (*Hint:* Use a graphing calculator to find the intersection of the graphs of $y = f'(S)$ and $y = 1$.)

Answers can be found on the next page.

Answers to **Harvesting Cod**

a. Replace a with e^r and b with r/P.
b. Shepherd: $a\,[1+(1-c)(S/b)^c]\,/\,[1+(S/b)^c]^2$; Ricker: $ae^{-bS}(1-bS)$; Beverton-Holt: $a/[1+(S/b)]^2$
c. a; a represents the slope of the graph of $f(S)$ at $S=0$.
d. 194,000 tons
e. 256,000 tons
f. f is a strictly increasing function of S.
g. 157,000 tons